Investigation 4.1

INTRODUCTION TO MAPS AND BLUEPRINTS

What does it mean for one picture to be a good copy of another? What kinds of information do scaled drawings, like maps and blueprints, provide? How can you test to see if one drawing is a good copy of another? In this section of the module, you will explore these questions and develop some tests for "well-scaled" drawings.

A SCALE OF ONE CITY

1 inch = 600 feet = 3-minute walk

The following page shows a map of downtown Seattle. The scale indicates that 1 inch on the map represents 600 feet of actual distance. It also says that an inch on the map represents about a three-minute walk.

1. What actual distance do the mapmakers claim you can walk in one minute? Is this reasonable?

2. Find the actual distance between the following locations (travel only on streets):

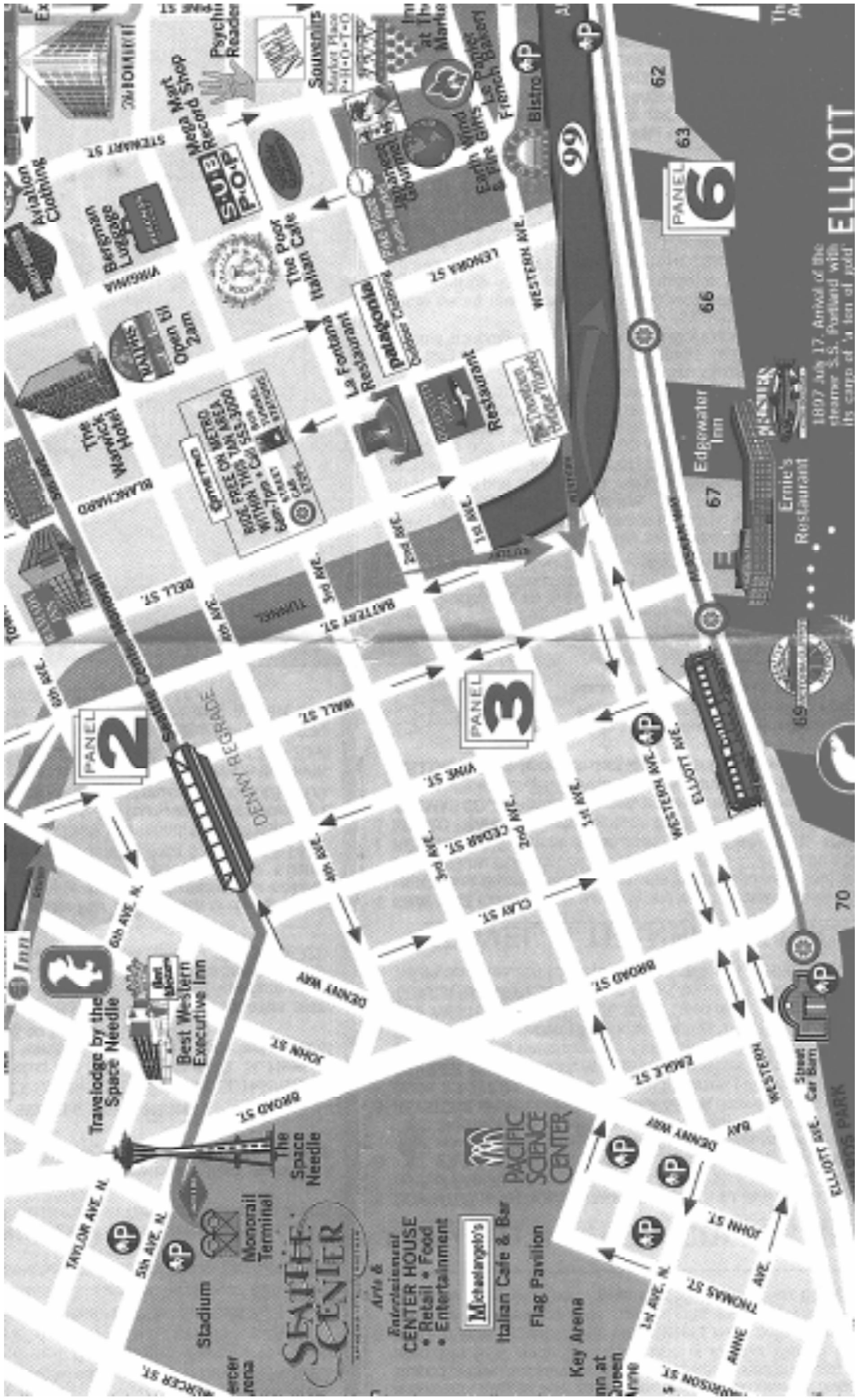
a. The Space Needle to the intersection of Broad St. and Western Ave.

b. Mega Mart Record Shop to the intersection of Eagle St. and 2nd Ave.

3. Lorena is at the Ramada Inn and has arranged to meet a friend at the intersection of Thomas St. and 1st Ave. N. Approximately how much time will her walk take?

FOR DISCUSSION

Other than maps, what pictures have you seen where the images have been scaled to a smaller or a larger size?


ELLIOTT
PANEL 6
PANEL 3
PANEL 2
DENNY REGRADE
SEATTLE CENTER
The Space Needle
Monorail Terminal
Stadium
PACIFIC SCIENCE CENTER
CENTER HOUSE
Flag Pavilion
Key Arena
Travelodge by the Space Needle
Best Western Executive Inn
The Warwick Hotel
Edgewater Inn
Ernie's Restaurant
Le Fontana Restaurant
The Poor Italian Cafe
Bergman Luggage
Aviation Clothing
Mega Mart Record Shop
Psychic Reader
WESTERN AVE.
ELLIOTT AVE.
1st AVE.
2nd AVE.
3rd AVE.
4th AVE.
5th AVE. N.
6th AVE. N.
TAYLOR AVE. N.
DENNY WAY
JOHN ST.
BROAD ST.
CLAY ST.
CEDAR ST.
VINE ST.
WALL ST.
BATTERY ST.
BELL ST.
BLANCHARD
LENORA ST.
VIRGINIA
STEWART ST.
THOMAS ST.
EAGLE ST.
70
69
67
66
63
62
99

WORKING WITH FLOOR PLANS

READING A BLUEPRINT

When architects draw plans for an apartment building, a house, a school, an office, a park, or a sports complex, they include many different sketches, each used for a different purpose. Here's an example of a *blueprint* for the outside of a house:

"Front elevation" is architect talk for "front of house."

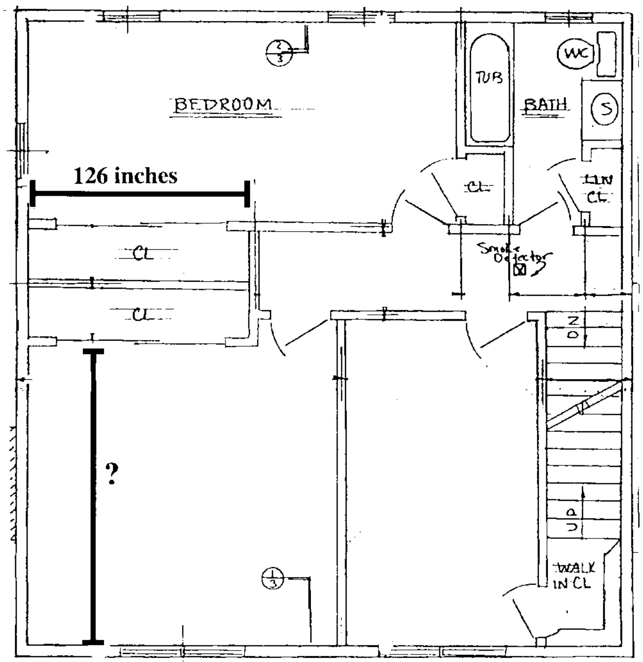
The architect has indicated on the blueprint that the distance from the top of the roofs to the top of the chimneys is exactly 3 feet.

4. Use the blueprint, a ruler, and a calculator to find the following measurements.

a. The total height of each chimney

b. The dimensions of the garage door

The next blueprint shows the floor plan of the second floor of the same house:



~SECOND FLOOR PLAN~

5. A measurement has been erased and replaced by “**?**” in this floor plan. Calculate this missing value.

6. Find the overall (full-size) dimensions of the entire second floor.

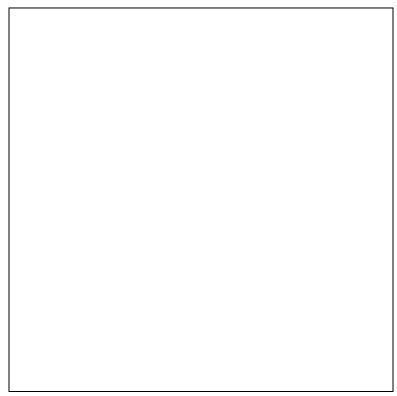
7. **Project** Make a floor plan for the house or apartment where you live or a building you visit frequently. Include measurements like the ones given in the floor plan above. Make sure everything is drawn to scale, including things like fixtures (sinks, bathtubs, and so on) and appliances.

Investigation 4.2

What Is a Scale Factor?

Most maps and blueprints provide a scale that allows you to calculate actual distances and lengths. Depending on the map, 1 inch might represent 1 mile (if the map is a detailed view of a small region) or 1 inch might represent 100 miles (if the map shows a larger region). The term *scale factor* describes what reduction or enlargement from the actual size was used to obtain the map, blueprint, or picture. In this investigation, you will develop a more precise definition of "scale factor" and examine how to calculate it.

1. Each side of this square has length 2 inches.



What do you think the following instruction means?

"Scale the square by a factor of $\frac{1}{2}$."

Draw a figure showing what you think it would look like to scale the square by $\frac{1}{2}$. If you can think of more than one way to interpret the statement, draw a separate figure for each idea.

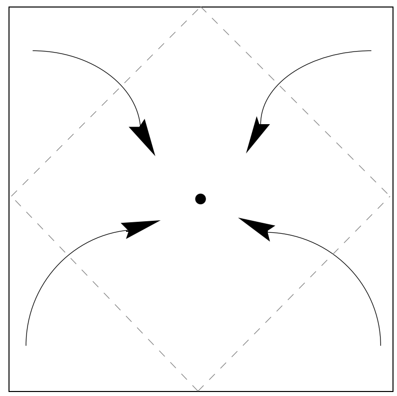
A Matter of Interpretation

Carlo and Amy have different interpretations of what it means to scale the square by a factor of $\frac{1}{2}$. See if you agree with either of their explanations.

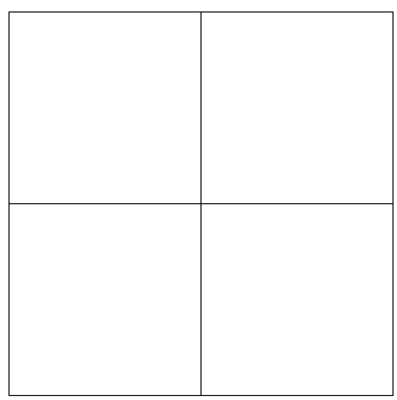
Carlo: When it said to scale by $\frac{1}{2}$, I drew a square that was half the size of the first one—you know, half the *area*. Since the area of the original square is $2 \times 2 = 4$ square inches, I needed to make a square with an area of 2 square

Why does Carlo's folding method work? How long are the sides of his new square?

inches. One neat way to do this is to fold all four corners of the square to the center.



Amy: I thought that scaling by half meant we were supposed to draw the *sides* half as long. The first square has sides that are 2 inches long, so the scaled one should have sides that are 1 inch long. I drew a horizontal and vertical line on the square to divide the length and width in half. This gives me four squares, each scaled by a factor of $\frac{1}{2}$.



In fact, there isn't just one correct way to interpret the phrase "scale by $\frac{1}{2}$." Words can mean different things to different people. But by convention, Amy's meaning of scaling is the one that most people use.

DEFINITION

Scaling a figure by a factor of r: When you scale a figure by a factor of r, your new figure will have lengths r times the corresponding lengths of the original figure.

The value of the scale factor, r, can be any positive number, including a fraction.

CHECKPOINT

2. What features of a square are invariant when you scale it by a factor of $\frac{1}{2}$?

3. A square has a sidelength of 12 inches. How long is each side of a new square created by scaling it by each of the following factors?

a. $\frac{1}{4}$

b. $\frac{1}{3}$

c. $\frac{2}{3}$

d. 2

e. 1

f. 1.3

4. If you scale a figure by the following values of r, will the new figure be smaller than, larger than, or the same size as the original one?

a. $r = \frac{3}{5}$

b. $r = 1$

c. $r = 3$

d. $r = 0.77$

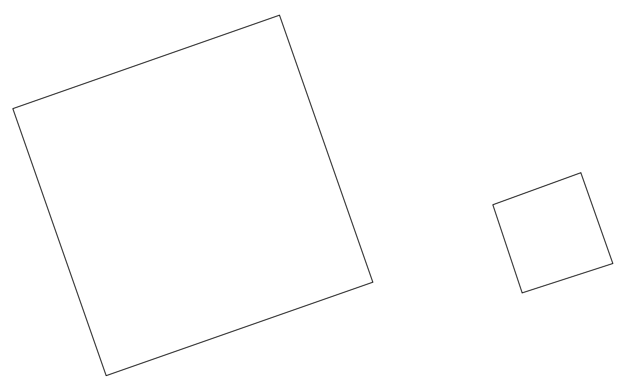
TAKE IT FURTHER

5. Suppose that you take a picture and scale it by a factor of $\frac{1}{2}$. You then take your *scaled* picture and scale *it* by $\frac{1}{4}$. By how much overall have you scaled the original picture?

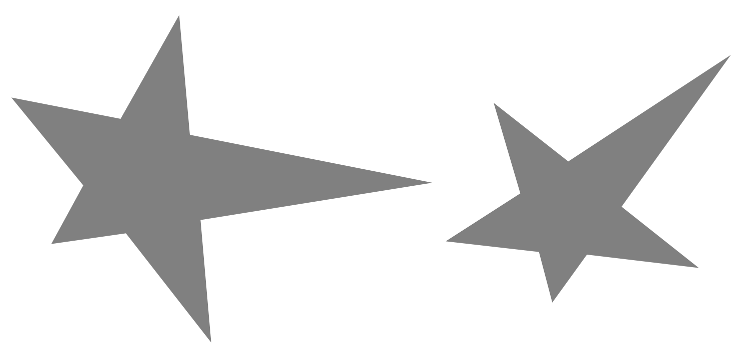
CALCULATING SCALE FACTORS

6. In each pair of pictures below, what scale factor will transform the picture on the left into the scaled picture on the right?

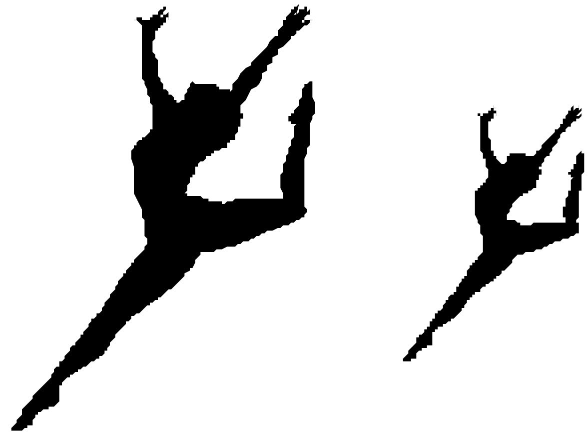
a.



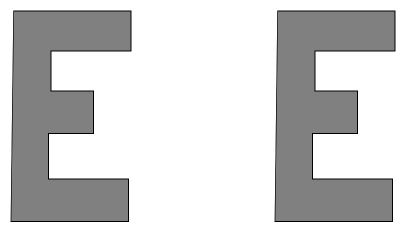
b.



c.



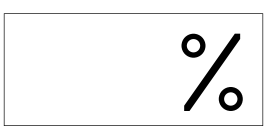
d.



7. Now suppose that the original pictures are those on the right. By what factor would you scale them to get the pictures on the left?

8. Compare your answers to Problem 7 with your answers to Problem 6. What is the relationship between them?

9. Many photocopy machines have a feature that allows you to reduce or enlarge (that is, to scale) a picture. Enter the amount “80%” (or some other percentage) on a photocopy machine and copy a picture. (Some machines require the factor as a decimal; for this example, the amount would be 0.80.)



Copier panel for entering a reduction or enlargement percentage

a. By what factor have you scaled the picture?

b. If you want to scale a picture by a factor of $\frac{3}{4}$, what percentage would you enter?

AREA AND VOLUME

These questions ask you to look at how the area or volume of a figure changes when you scale it.

10. **a.** Draw a square with 1-inch sides and scale it by a factor of 2. How many copies of the 1-inch square fit inside the scaled square?

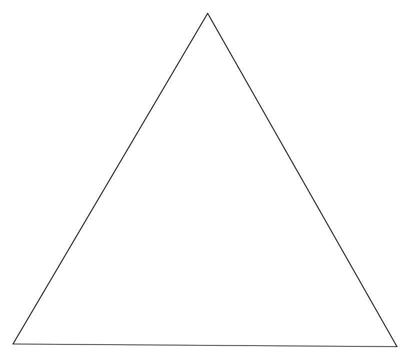
b. Start with a 1-inch square again, and scale it by a factor of 3. How many copies of the 1-inch square fit inside the scaled square?

c. If you scale a 1-inch square by a positive integer r, how many copies of it will fit inside the scaled square?

11. The equilateral triangle below has 2-inch sides.

a. Draw a scaled version of the triangle, using a factor of $\frac{1}{2}$. How many of these scaled triangles can you fit inside the original triangle?

b. Draw a scaled version of the triangle, using a factor of $\frac{1}{3}$. How many of these scaled triangles can you fit inside the original triangle?



12. A cube has edges of length 1 inch.

a. If the cube is scaled by a factor of 2, how long will the sides of the new cube be? How many copies of the original cube will fit inside the scaled cube?

b. If the original cube is scaled by a factor of 3, how long will the sides of the new cube be? How many copies of the original cube will fit inside the scaled cube?

c. If you scale the original cube by a positive integer r, how many copies of the original cube will fit inside the scaled cube?

Investigation 4.3

WORKING WITH DIRECTIONS: A LOGO ACTIVITY

When drawing a picture using the computer language Logo, people sometimes decide that their final picture is either too big or too small. This investigation looks at what adjustments need to be made in a set of Logo instructions in order to scale a picture by any amount.

1. a. Try this Logo procedure, which draws a regular pentagon:

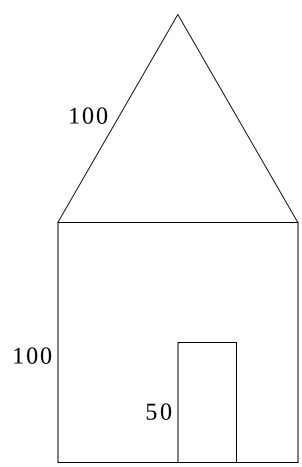
```
to Pent
  repeat 5 [fd 100 rt 72]
end
```

b. Modify the procedure so that it draws the pentagon scaled by $\frac{1}{2}$. Then modify it to draw the pentagon scaled by 3.

2. Write a Logo procedure that will draw your initials. Then modify it to draw your initials again, scaling the first drawing by a factor of 2.

SCALING A HOUSE

Here's a picture of a house:



3. Write a Logo procedure that will draw this house.

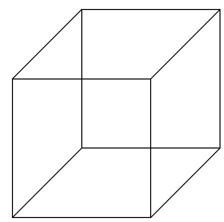
4. Write a Logo procedure that will draw this house scaled by a factor of 2.

5. Write a Logo procedure that will draw this house scaled by $\frac{1}{2}$.

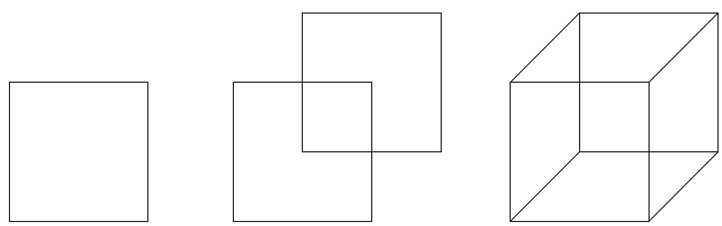
6. Write *one* house-drawing procedure that will produce a house at *any* scale (that is, using *any* scale factor). Your procedure should take one input (the scale factor). When the scale factor is 1, you get the house in Problem 3. When it's 2, you get the house in Problem 4; when it's $\frac{1}{2}$, you get the house in Problem 5.

TAKE IT FURTHER. . . . .

Here is a two-dimensional picture of a three-dimensional cube:



The picture was made by drawing a square, drawing another square with one vertex at the center of the original one, and then connecting corresponding vertices:



(1) (2) (3)

How long are the "diagonal" connecting lines?

7. Write a Logo procedure that will produce this drawing (make the lengths of the edges of the cube a "nice" number, like 100).

8. What happens to your cube procedure if you *increase* each length by 20? Explain.

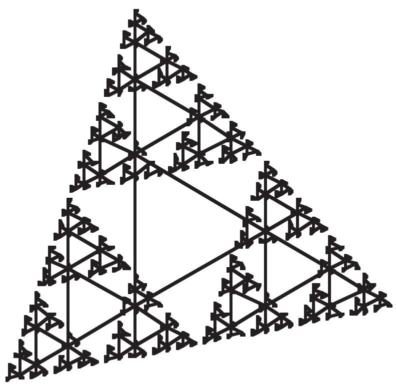
9. Write a Logo procedure that will draw a cube at twice the scale of the one in Problem 7 (that is, double the scale factor that would be needed to draw the cube in Problem 7).

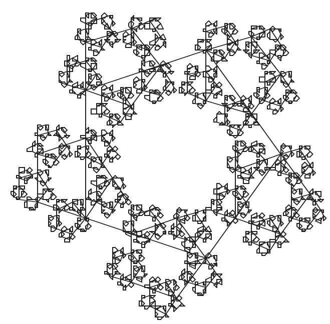
10. Write a Logo procedure that will draw a cube at $\frac{1}{3}$ the scale of the one in Problem 7.

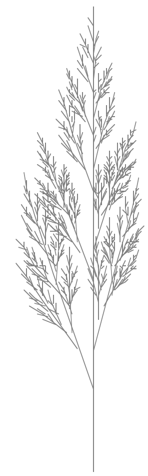
11. Write *one* cube-drawing procedure that will produce a cube at *any* scale. As in Problem 6, your procedure should take a scale factor as an input.

Self-similar **pictures are ones whose parts are (more or less) scaled-down copies of the whole picture.**

The branch of geometry known as "fractal geometry" was developed to help people analyze complex shapes like tree-shapes, cloud-shapes, and mountain-shapes. A new idea called "self-similarity" plays an important role in fractal geometry. Here are several examples of self-similar designs you can create easily with Logo:







12. Try each one of the following procedures by itself.

```
to Pent0
  repeat 5 [fd 40     rt 72]
end

to Pent1
  repeat 5 [fd 40 * 0.618     rt 72]
end

to Pent2
  repeat 5 [fd 40 * 0.618 * 0.618     rt 72]
end
```

a. If Pent0 is "regular size," then at what scale is Pent1? And suppose that Pent1 is the "regular size"—at what scale is Pent0? Compare Pent0 to Pent2 in the same way.

b. Now comes the fun. Change Pent0 in the following way:

```
to Pent0
  repeat 5 [fd 40   Pent1   rt 72]
end
```

Carefully describe what you see.

c. Now change Pent1 in a similar way. Put the instruction Pent2 just before rt 72, as you had put Pent1 just before rt 72 in Pent0. Carefully describe what you see.

There are many ways of embedding smaller figures. Does repeat 3 [fd 80 tri1 rt 120] give the same picture as repeat 3 [fd 80 rt 120 tri1]?

13. Try the same kind of experiment with triangles. Create Tri0 to draw an equilateral triangle with sides 80 steps long. Let each smaller triangle be a scaled version of the larger by 0.5, instead of 0.618.

The "Bud" procedure draws just a point. We use it so that all of the "Tree" procedures that follow will have the same structure.

14. This time, let's begin small and work up. Make Bud, Tree1, and Tree2 like this:

```
to Bud
 forward 0   back 0
end
```

```
to Tree1                  to Tree2
  fd 2                      fd 4
  lt 25 bud rt 25           lt 25 bud rt 25
  rt 25 bud lt 25           rt 25 bud lt 25
  rt 5                      rt 5
  fd 2                      fd 4
  rt 5 bud lt 5             rt 5 bud lt 5
  bk 2                      bk 4
  lt 5                      lt 5
  bk 2                      bk 4
end                       end
```

***Scaled-up* and *scaled-down* are shorthand words for "scaled by a factor greater than 1" and "scaled by a factor less than 1."**

a. Try all three procedures. Sketch and describe what each does. (You may need to sketch at a larger scale!) Tree2 is a scaled-up copy of Tree1—by what scale factor?

b. Edit Tree2, and *replace* each bud with the instruction Tree1, like this:

```
to Tree2
 fd 4
   lt 25   tree1   rt 25
   rt 25   tree1   lt 25
 rt 5
 fd 4
   rt 5    tree1   lt 5
 bk 4
 lt 5
 bk 4
end
```

Describe what has changed, and how it has changed.

Why did we use quotation marks around "twice the scale"? That is, in what way is Tree3 not just a double-sized Tree2?

c. Now carry this a step further. Create a Tree3 just like Tree2, but at "twice the scale" and using Tree2 instead of Tree1 for branches, as shown below.

```
to Tree3
 fd 8
   lt 25   tree2   rt 25
   rt 25   tree2   lt 25
 rt 5
 fd 8
   rt 5    tree2   lt 5
 bk 8
 lt 5
 bk 8
end
```

d. Following the same pattern, take this process through three more steps. Tree4 is shown for you.

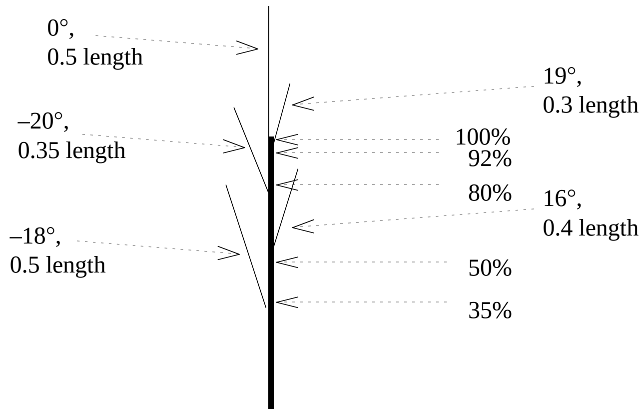
```
to Tree4              to Tree5   to Tree6
fd 16
lt 25 tree3 rt 25        .          .
rt 25 tree3 lt 25        .          .
rt 5                     .          .
fd 16
rt 5 tree3 lt 5
bk 16
lt 5
bk 16
end                     end        end
```

If you ignore the details of the branching, you can see that the “trunk” of the previous tree was quite simple. For example, look at Tree4.

```
to Tree4
 fd 16
  . . .                   some branching here
  . . .                   some branching here
 rt 5
 fd 16
  . . .                   some branching here
 bk 16
 lt 5
 bk 16
end
```

The trunk is in two congruent sections, with a 5° bend between them to make the tree more graceful. Buds sprout in only three places: two at the bend, each angled off 25° from the trunk below them, and one at the very top, angled 5° to the right. No matter where the bud is on the tree, when a bud becomes a branch, the branch is exactly half the scale of the tree on which it sits.

The tree shown at the beginning of this section was developed from a slightly more complex plan, which was based (very roughly) on a twig someone actually found on the ground. A sketch of that plan is shown below.



If you think of the heavy line as the trunk, you will see that this trunk is not bent. (The real twig was bent, but the bend was ignored to make the programming easier.) You can see that "buds" sprouted in more than three places, and no two sprouted in the same place. Also, they grew to different scales, some as long as half the size of the trunk, and some less than a third the size of the trunk.

15. a. How many limbs grow from the trunk shown above? How far up the trunk is the lowest limb? What angle does that limb make with the trunk? How long is that limb, compared to the trunk? How far up the trunk is the highest limb? How do its size and angle compare to the trunk?

b. Part of a procedure that follows this plan is shown below. The input is convenient because it allows you to draw the basic "skeleton" at various sizes. Finish designing this procedure.

```
to Vtree4  :trunksize
 fd 0.35  * :trunksize
   lt 18   branch 0.5 * :trunksize   rt 18
 fd 0.15  * :trunksize
   rt 16   branch 0.4 * :trunksize   lt 16
     ⋮
   rt 19  . . .  lt 19
 fd 0.08   * :trunksize
   lt 0   branch 0.5 * :trunksize   rt 0
 bk :trunksize
end
```

To test Vtree4, you will need to create Branch, which just draws a branch of the right length.

```
to Branch :length
 fd :length  bk :length
end
```

c. Now try each procedure with various inputs. For example, you might try Branch 10, Branch 50, Vtree4 100, and Vtree4 50. Describe the results, using the idea of scale.

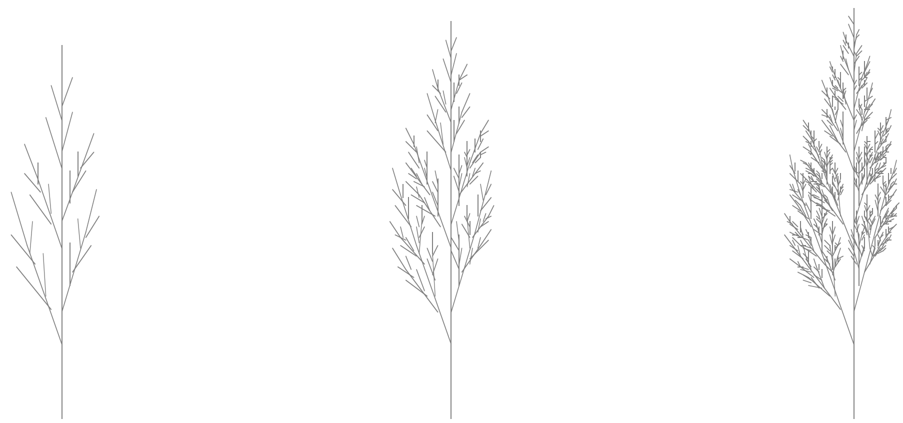
16. The following makes a very lovely tree when fully developed.

a. First, create Vtree3, identical to Vtree4: no change except the title.

b. Then edit Vtree4 and replace Branch with Vtree3 everywhere. Try out Vtree4 as you did before. What has changed?

c. If you create Vtree2 absolutely identical to Vtree3 (no change except the title), you can then edit Vtree3 and replace all its Branches with Vtree2s. Try this.

d. Continue the process another step.

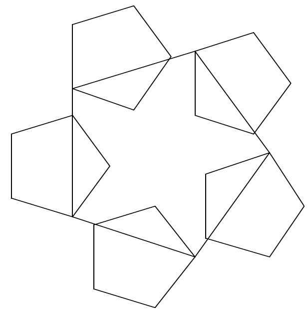


The simplest picture above shows the branching when each limb of the original tree is replaced by a copy of that tree. The copies are scaled by different factors so that the height of *each* trunk matches the length of the limb that it replaces. If you then replace the branches in the scaled-down trees with further scaled-down trees, each branch will sprout five twigs, as in the middle picture above. Still another replacement, with five leaves per twig, is very tree-like.

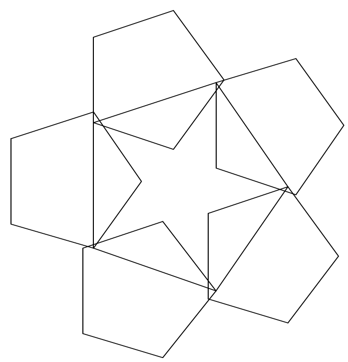
WAYS TO THINK ABOUT IT

To draw trees, you could choose appropriate scale factors either by measuring an actual tree (or twig) or by experimenting and choosing something pleasing. But how did we come up with a scale factor like 0.618 used in Problem 12? We determined the scale factor partly by experimenting and partly by thinking about the geometry.

By experimenting, you might discover that pentagons scaled by $\frac{1}{2}$ look like this:

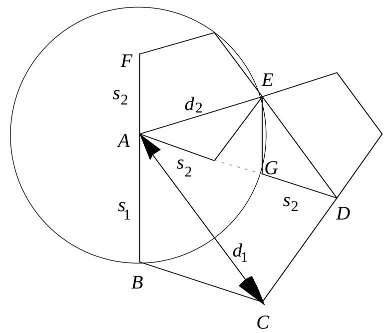


That might suggest trying to get the little ones to "fit." It requires a larger scale factor: perhaps $\frac{2}{3}$ instead of $\frac{1}{2}$? Well, no, that produces something too big:



Between 0.5 and 0.667, you might try 0.6. To the eye, it is pretty hard to tell that there's anything better. But, in fact, there is! One gets used to asking things like "What number that has a lot to do with pentagons

is close to 0.6?" Here's a picture from which it is possible to figure out the exact value.



The large pentagon $ABCDE$ has sides (like $\overline{CD}$) of length s_1 and diagonals (like $\overline{AC}$) of length d_1. The scaled-down pentagons have sides (like $\overline{AF}$) of length s_2 and diagonals (like $\overline{AE}$) of length d_2. A circle centered at A has radius s_1; notice that it also has radius d_2.

Now you can start writing down relationships that you see. The scale factor that you're looking for (call it r for ratio) scales s_1 to s_2 and d_1 to d_2. Algebraically, you can write it this way:

$$r = \frac{s_2}{s_1} = \frac{d_2}{d_1}.$$

And because the scale was deliberately adjusted in order to get $d_2 = s_1$, you can also write $r = \frac{s_1}{d_1}$. To summarize,

$$r = \frac{s_2}{s_1} = \frac{d_2}{d_1} = \frac{s_1}{d_1}.$$

17. **a.** The diagram seems to show that $d_1 - d_2 = s_2$. Explain.

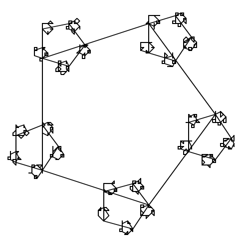
b. From the relationship $\frac{s_2}{s_1} = \frac{s_1}{d_1}$, you can derive another relationship: $s_1{}^2 = \ldots$. Complete the algebra.

c. Divide both sides of your new equation by $d_1^{\,2}$. Now, using the fact that $r = \frac{s_1}{d_1}$, write an equation that contains only r^2, s_2, and d_1.

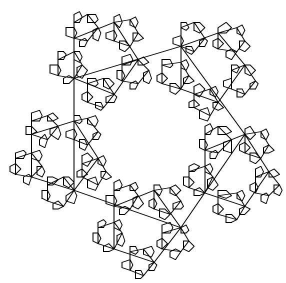
d. Now, "get rid of" s_2 by expressing it in terms of d_1 and d_2, and "get rid of" d_2 by expressing it in terms of d_1 and r. Finally, "get rid of" d_1 by factoring it out and writing the fraction in lowest terms. The remaining equation is in r alone and can be solved exactly for r. The value of r is known as the "golden ratio."

Find out about the golden ratio. Along with many interesting mathematical connections, a great deal of silly stuff has been written about this number. But some of the silly stuff is interesting too, even if often wrong!

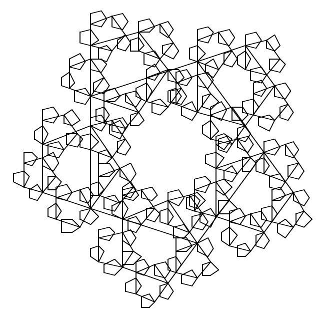
The ratio r^2 is valuable, too. Look at the following pictures. One seems sparse, one seems crowded and jumbled, and one seems "just right."



$r = \left(\frac{1}{2}\right)^2$



$r = (0.618\ldots)^2$



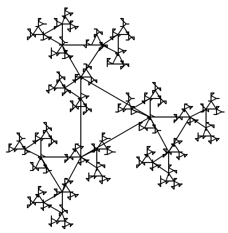
$r = \left(\frac{2}{3}\right)^2$

18. Try to create the lace-like pentagon design that is shown on page 14.

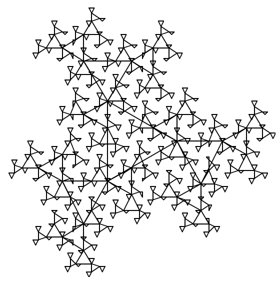
19. Here's a new twist on making a design with triangles. Like `TriO`, `Twist` draws a triangle, with a scaled-down triangle at each of its corners. But, instead of the smaller triangle growing in the same direction, it is twisted 30° to the left:

```
To Twist
 repeat 3 [fd 80  lt 30  twist1  rt 30  rt 120]
end
```

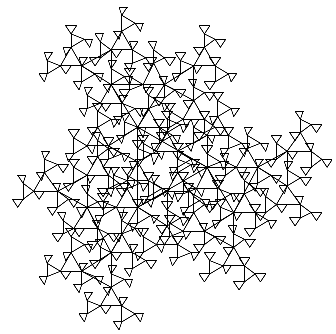
These pictures show three different scale factors. Again, one is too sparse to be really elegant, one too crowded, and one just right.



$r = \frac{1}{2}$



$r = ?$



$r = 0.65$

Find the ratio used in the middle figure. You may either start by experimenting to get close, and *then* examine the geometry, or you may start directly with the geometry.

Why do we constantly say "more or less"?

In geometry, *self-similar* pictures are ones whose parts are (more or less) scaled-down copies of the whole pictures. For example, `Pent0` drew a picture that contained five `Pent1`s, which were each (more or less), smaller versions of `Pent0`. Likewise, the branches on the tree were each (more or less) just smaller versions of the tree itself. Another useful way of looking at it: if you magnify just one of the small parts, you see (more or less) the original picture.

The same idea occurs outside of geometry as well.

20. The repeating decimal 0.33333... contains a scaled-down copy of itself. It can be written as the terminating decimal 0.3, plus a copy of the repeating decimal scaled down by a factor of $\frac{1}{10}$:

$$0.33333\ldots = 0.3 + \left(\frac{1}{10} \times 0.33333\ldots\right)$$

Or it can be scaled up (by a factor of 10); the new, larger number contains the original:

$$10 \times 0.33333\ldots = 3.33333\ldots = 3 + 0.33333\ldots$$

In other words, the original number times 10 is the same as the original number plus 3. Explain how this relationship can be used to convert 0.33333 into a rational number (a fraction).

When the terms in a sequence are all related by a single scale factor, the sequence is called *geometric*. The sum of such a sequence is called a *geometric series*.

21. You just showed how to find the sum of an infinite number of fractions (as long as they were all related by a single scale factor). The fractions you added were:

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots .$$

a. Find a suitable scale factor for scaling each of the following "geometric series" up or down to show how they are "self-similar."

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{3^6} + \frac{1}{3^7} + \frac{1}{3^8} + \cdots$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \cdots$$

b. Now "find the sum"; that is, find a single rational number (an integer or fraction) that has the same value.

22. Here are two more calculations that seem to involve a kind of self-similarity but do not involve any "scaling." See what you can do to convert them into expressions that you can evaluate. (The values are ones you've used earlier in this investigation.)

Expressions of this form are called "continued fractions" and are useful mathematical structures.

a.

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}$$

b.

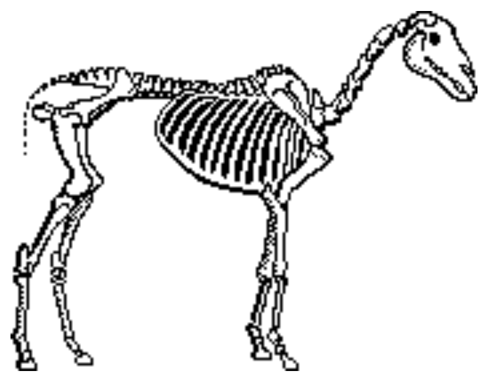
$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}}$$

Investigation 4.4

WHAT IS A WELL-SCALED DRAWING?

The maps and blueprints in Investigation 4.1 are all well-scaled drawings of cities and houses. Sometimes, though, a reduction or enlargement is not a good copy of the original. If you look at yourself in a funhouse mirror, you'll see that your body has been scaled in a very bad way indeed—the mirror might stretch you to appear as skinny as a matchstick, or shrink you to appear one foot tall!

Take a look at this picture of a horse skeleton:

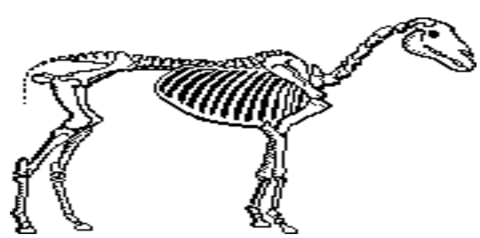


The original horse skeleton

Some computer drawing programs have scaling features that allow you to scale separately in the horizontal and vertical directions. The result is a drawing that is "poorly-scaled" or distorted.

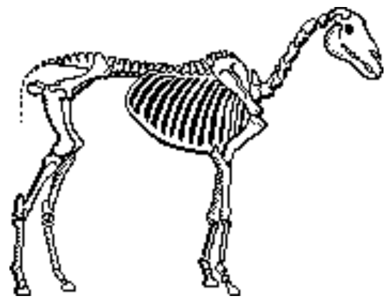
Since we don't have a funhouse mirror, we decided to play with the scaling features of a computer drawing program instead. Sometimes we made well-scaled copies of the original skeleton; other times, we distorted it. Which, if any, of these four pictures are well-scaled copies of the original?

1



2

3



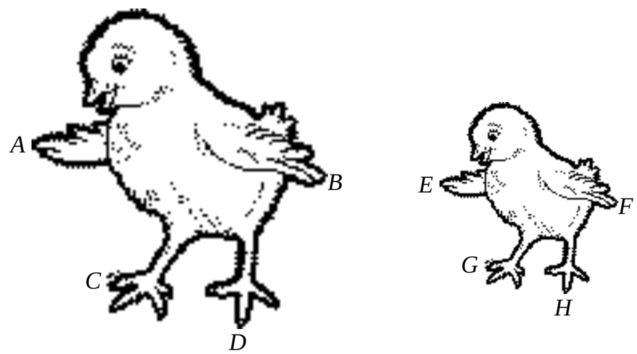
4

1. Decide which of the four pictures are accurate enlargements or reductions of the original horse skeleton.

FOR DISCUSSION

Share your answers to the horse skeleton question with your classmates. What characteristics of the horse skeleton drawings helped you to make your decision? Did your classmates pay attention to different features?

2. The picture below shows two baby chicks labeled with points A through H:

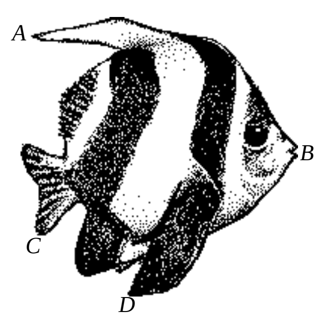


Here are the distances between some of the points:

$$AB = 4 \text{ cm}, \; CD = 2 \text{ cm}, \; EF = 2.5 \text{ cm}, \; GH = 1.25 \text{ cm}.$$

How can you use these measurements to help convince someone that the two chicks are well-scaled copies of each other?

3. Four locations—A, B, C, and D—are labeled on the fish below:



WHAT IS A WELL-SCALED DRAWING?
(continued)

Joan measures the distance between A and B, as well as between C and D. She calculates that $\frac{AB}{CD} = 2.6$. Michael sees her answer and asks, "But what's your unit? Is the answer 2.6 centimeters, 2.6 inches, 2.6 feet, or something else?"

How would you respond to Michael's question?

Investigation 4.5

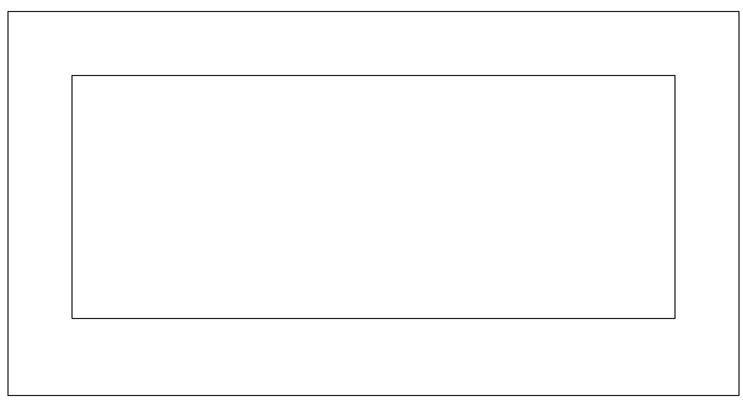
Testing for Scale

In the last investigation, you thought about which characteristics of horse skeleton pictures did (or did not) make them well-scaled copies of each other. Now you'll do the same for pairs of simple geometric figures like rectangles, triangles, and polygons.

Mathematicians usually use "scaled," rather than "well-scaled" to refer to proportional figures. Also, the term "scale drawing" is used in architecture and other fields. From here on, we will use "scaled" rather than "well-scaled"; both terms have the same meaning.

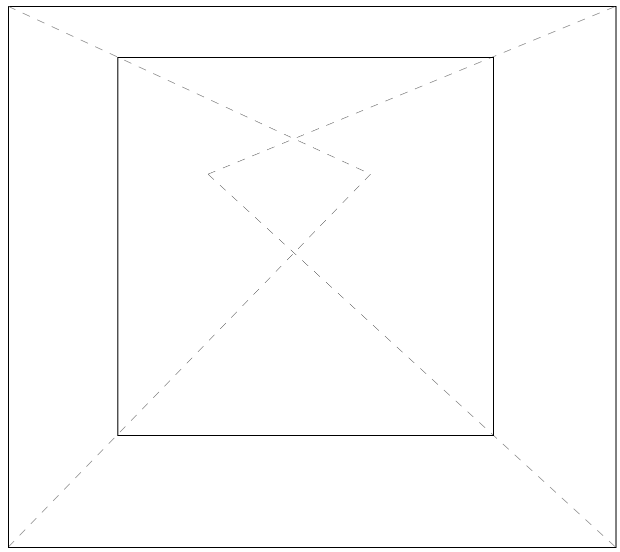
Scaled Rectangles

1. Take whatever measurements and do whatever calculations are necessary to check whether the two rectangles below are scaled copies of each other.

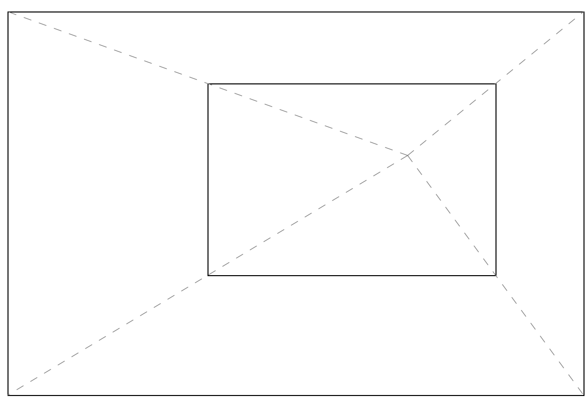


2. Here are three more pairs of rectangles. In which pair(s) are the rectangles scaled copies?

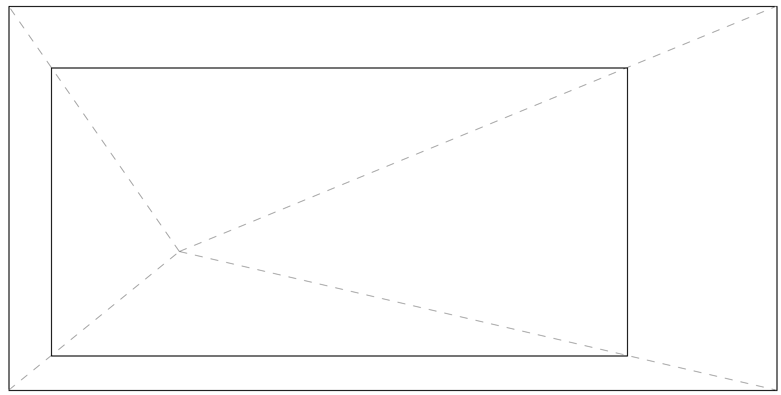
a.



b.



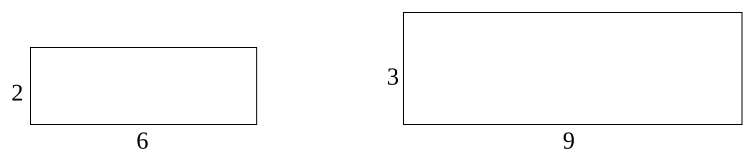
c.



Hmm . . . what are all these dashed lines doing here?

FOR DISCUSSION

Here are two rectangles:



- How can you tell that the rectangles are scaled copies?
- Some people use the phrase *corresponding sides* when talking about scaled copies. What do you think this means?
- Some people say that the corresponding sides of these rectangles are *proportional*. What do you think this means?

3. Here are the length and width measurements of seven rectangles. Match the rectangles that are scaled copies of each other. How did you make your decision?

a. $4'' \times 1''$

b. $3'' \times 2''$

c. 10″ × 5″

d. 4″ × 6″

e. 5″ × 3″

f. 16″ × 4″

g. 8″ × 4″

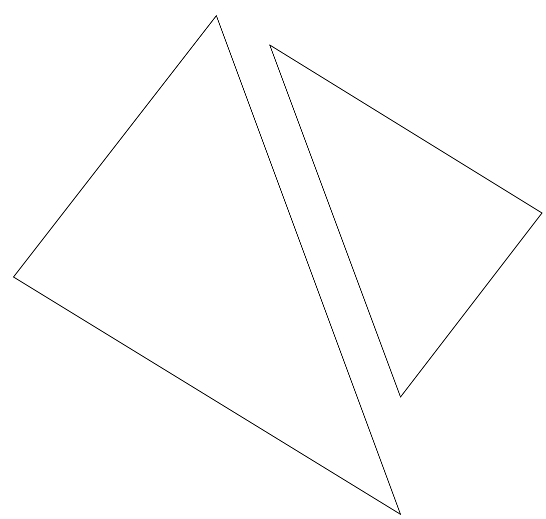
A *ratio* is the quotient of two numbers. If only one number is given, the second number is understood to be 1. "The ratio is 1.5" is a shorthand way to say "the ratio is 1.5 to 1," which is equivalent to "the ratio is 3 to 2."

4. a. The ratio of length to width of a particular rectangle is 1.5. A scaled copy has a width of 6. What is the length of the scaled copy?

b. Two rectangles are scaled copies of each other. The ratio of the length of one rectangle to the length of the other is 0.6. If the smaller rectangle has a width of 3, what is the width of the larger one?

SCALED TRIANGLES

5. Take whatever measurements and do whatever calculations are necessary to check whether the two triangles below are scaled copies of each other. Explain your method.



6. Kaori has two triangles with equal angle measurements. The sides of one triangle are 4, 6, and 8, and the sides of the other are 9, 6, and 12. She says that the triangles are not scaled copies because

$$\frac{4}{9} = 0.44\ldots,$$
$$\frac{6}{6} = 1,$$

and

$$\frac{8}{12} = 0.66\ldots.$$

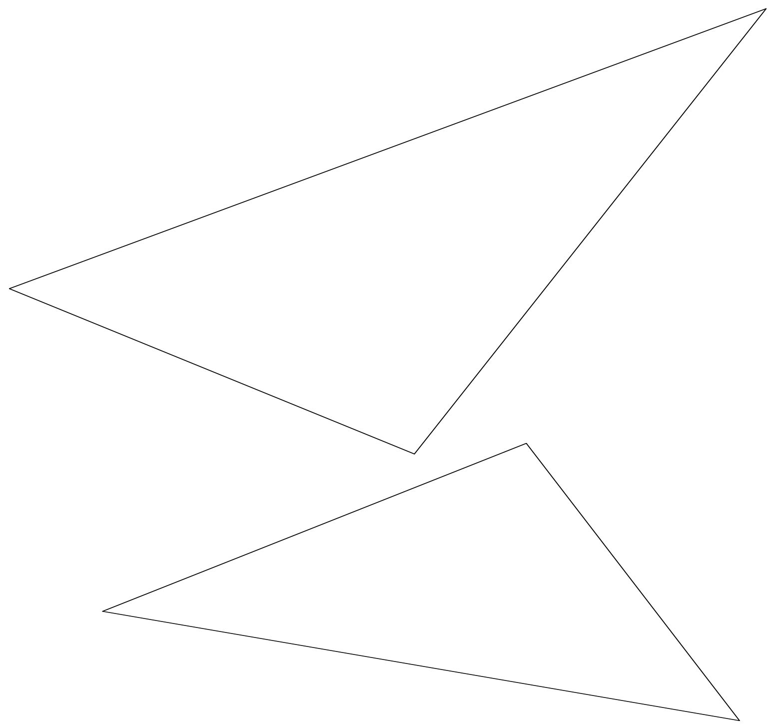
Do you agree?

Checking For Scaled Triangles Without Measuring

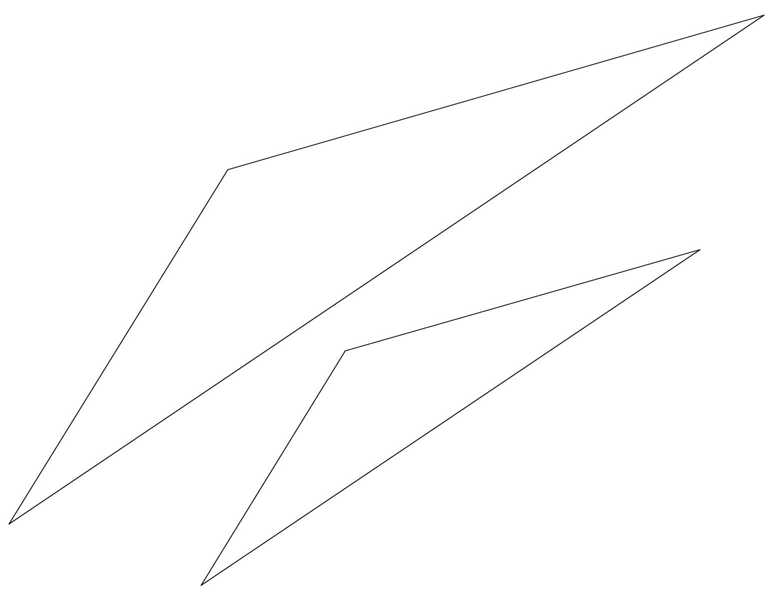
Think back for a minute to congruent triangles. How did you determine whether two triangles were congruent *without* taking any measurements or calculations? One way is to cut out the triangles and lay them on top of each other. If the triangles can be arranged so that they perfectly coincide, then they are congruent. Perhaps there's also a visual way to test whether two triangles are scaled copies of each other without having to take any measurements.

7. In each of the pairs of triangles on the following page, the triangles are scaled copies of each other. Trace and cut out each pair. Then play with the triangles and look for some visual clues that might make good tests for recognizing scaled triangles. For instance, if you place two corresponding angles on top of each other, what do you notice about the triangles' corresponding sides?

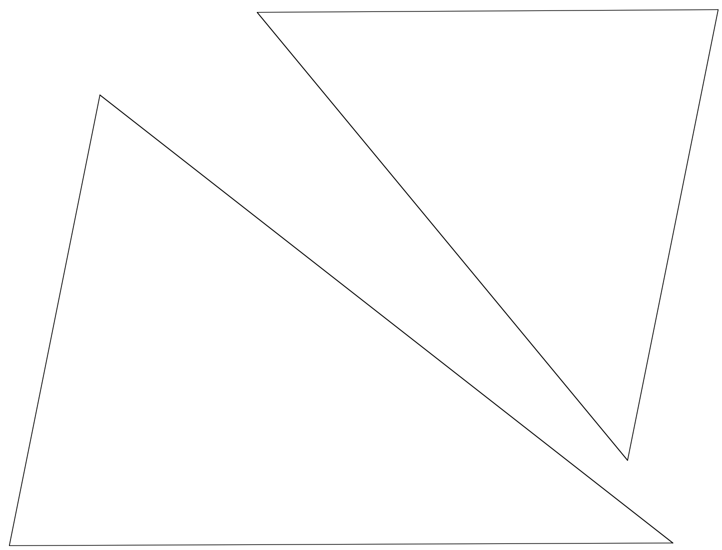
a.



b.



8. Trace and cut out the pair of triangles below. Based on the observations you've made in the previous problem, do you think these triangles are scaled copies of each other?



FOR DISCUSSION

You've now tried two approaches for checking whether triangles are scaled copies: 1) you measured various parts of the triangles and did some calculations, and 2) you relied strictly on visual aspects without taking any measurements.

Discuss the various measurements, calculations, and observations you made with your classmates. Did they pay attention to the same characteristics of the triangles that you did? Make a list of all the different approaches that you and your classmates suggest.

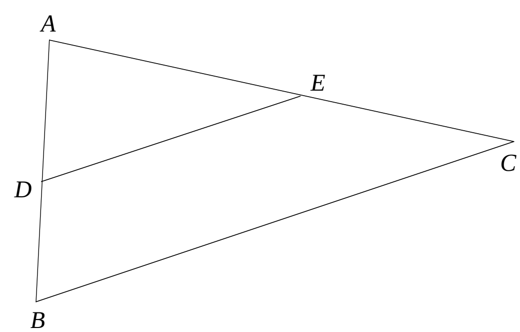
9. **Write and Reflect** Complete the following statement in as many different ways as possible:

> To test if two triangles are scaled copies of each other, . . .

For now, write every possible test that you think might work. You will return to scaled triangles later in this module and develop ways of proving (or disproving) these scaling tests.

10. **a.** In one classroom, students suggested that if the angles of one triangle are congruent to the corresponding angles of another triangle, then the two triangles must be scaled copies. Do you agree? Explore this conjecture with some triangles.

b. In the figure below, $\overline{DE} \parallel \overline{BC}$. Explain why, *if* the conjecture from part a is true, then this would prove that $\triangle ADE$ is a scaled copy of $\triangle ABC$.



c. Martin thinks that the angle test for triangles might also be a good way to check whether other polygons (like rectangles) are scaled copies. If the angles of one polygon are congruent to the corresponding angles of another, are the polygons scaled copies? Explain.

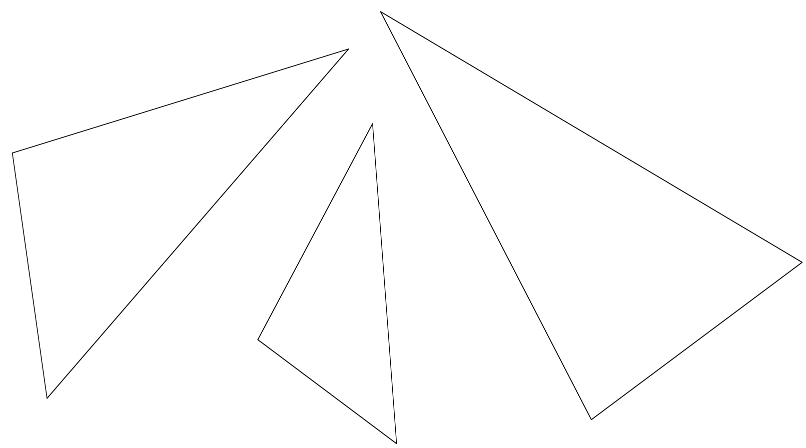
11. Sheena has an idea for how to make a scaled copy of a triangle:

> "Measure the sides of the triangle. Then add the same constant value (like 1, 2, or 3) to each side length. Draw a triangle with these new sidelengths."

Will this new triangle be a scaled copy of the original? Try it!

CHECKPOINT

12. Trace and cut out the triangles below. Use any of the methods your class has discussed to figure out whether any of the triangles are scaled copies of each other.



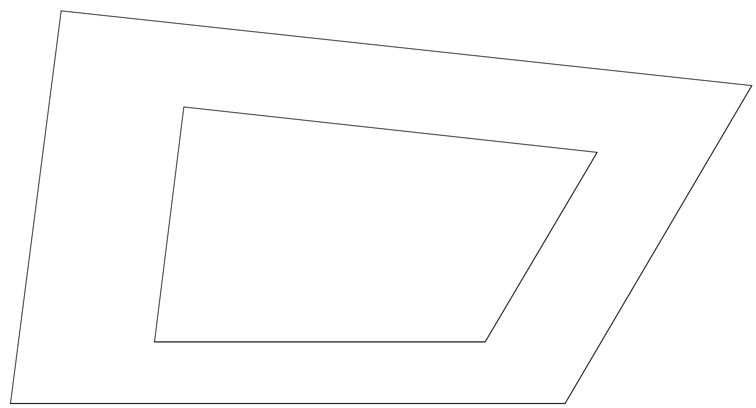
13. Two angles of one triangle measure 28° and 31°. Another triangle has two angles that measure 117° and 31°. Are the triangles scaled copies? How can you tell?

14. One triangle has sidelengths of 21, 15, and 18. Another triangle has side lengths of 12, 14, and 16. Are these triangles scaled copies? How can you tell?

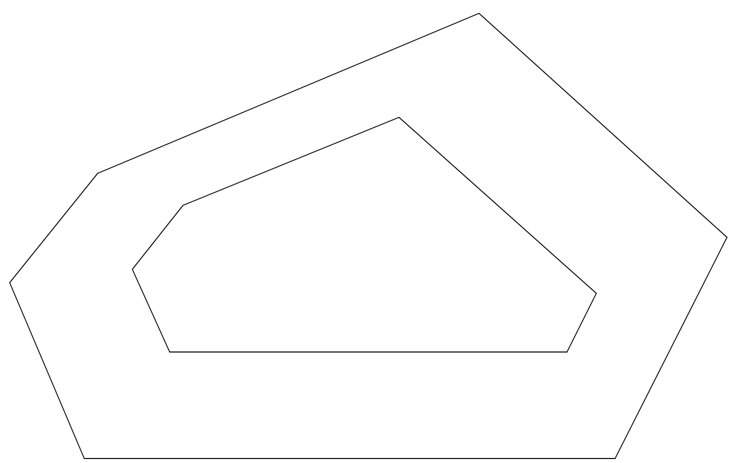
SCALED POLYGONS

15. In each figure below, decide if the pair of polygons are scaled copies. Explain how you made your decision.

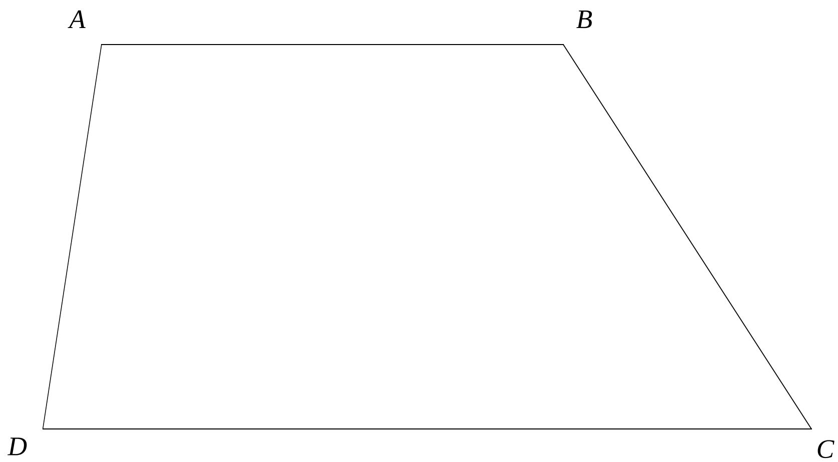
a.



b.



16. Trace the trapezoid below. Then draw another trapezoid inside of it that's a scaled copy (you choose the scale factor). How did you do it?



17. Draw two quadrilaterals so that the sides of one are twice as long as the corresponding sides of the other but neither is a scaled copy of the other.

18. Is either of the two statements below a valid test for checking whether two polygons are scaled copies?

> *Two polygons are scaled copies of each other if they can be arranged in some position so that their corresponding angles all have equal measures.*

> *Two polygons are scaled copies of each other if they can be arranged in some position so that their corresponding sides are all in the same ratio.*

Explain your answers. If these are not valid tests, what additional requirement(s) can you add to make a test that works?

19. A polygon is scaled by a factor of $\frac{3}{4}$. The original polygon is then compared to the scaled one. Find the ratio of

a. any two corresponding sides of the polygons;

b. any two corresponding angles of the polygons.

20. For each pair of figures listed below, explain why they must be scaled copies of each other, or give a counterexample to show why they are not always scaled copies.

a. any two quadrilaterals

b. any two squares

c. any two quadrilaterals with corresponding angle measurements equal

d. any two triangles

e. any two isosceles triangles

f. any two isosceles right triangles

g. any two equilateral triangles

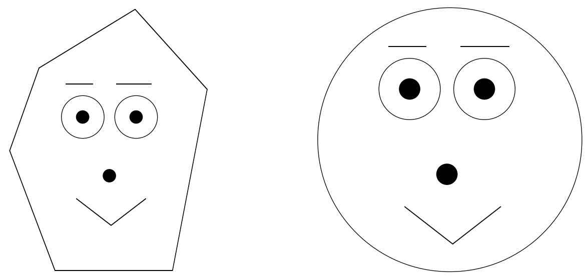
h. any two rhombi

i. any two regular polygons with the same number of sides

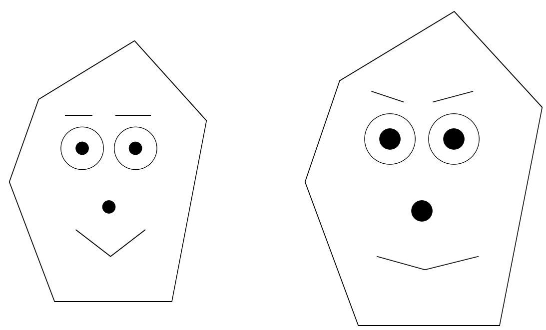
Investigation 4.6

THE MANY FACES OF SCALING

The pairs of faces below can't stay still long enough to remain scaled copies. Underneath each pair, there is a description of a scaling requirement they do not follow.

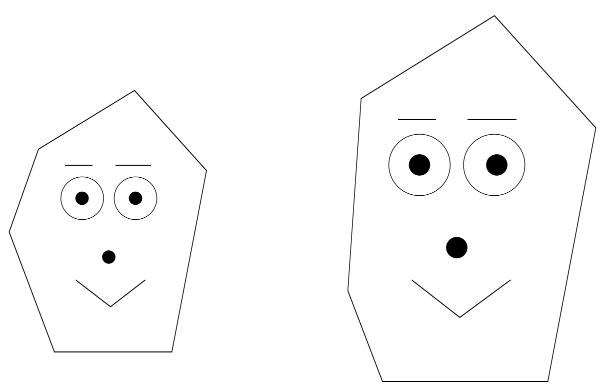


In scale drawings, figures might be a different size, *but must have the same* shape.



In scale drawings, corresponding angle measures *must be the same.*

Identify two ratios that prove these faces are not scaled copies.

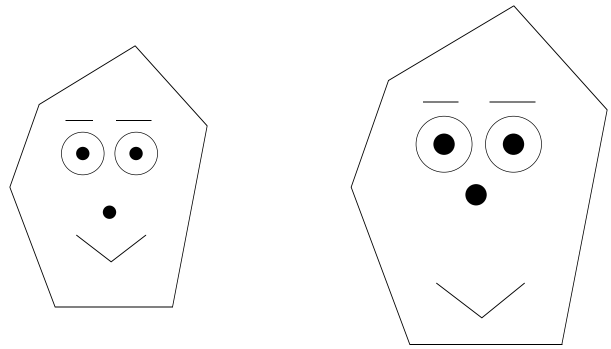


In scale drawings, any two pairs of corresponding lengths *must have the same ratio.*

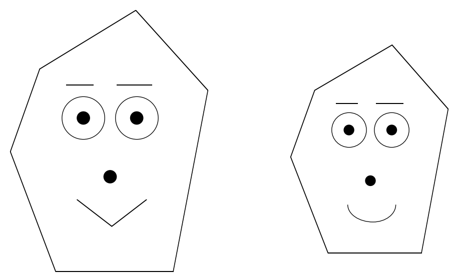
Investigation 4.6

THE MANY FACES OF SCALING (continued)

Identify two ratios that prove these faces are not scaled copies.



In scale drawings, the ratio between any two distances within *one drawing must be the same within the other drawing.*



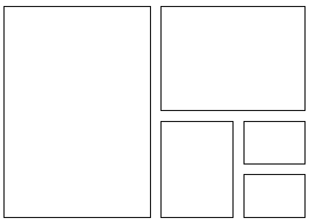
In scale drawings, straight line segments must remain straight.

1. Draw a whole new series of pictures (other than faces) to illustrate all of the scaling requirements shown above.

Investigation 4.7

RECTANGLE DIAGONALS

Testing for scaled copies often requires measurements and calculations, but in some special cases, there are ways to tell with no measurements at all. Here is one:

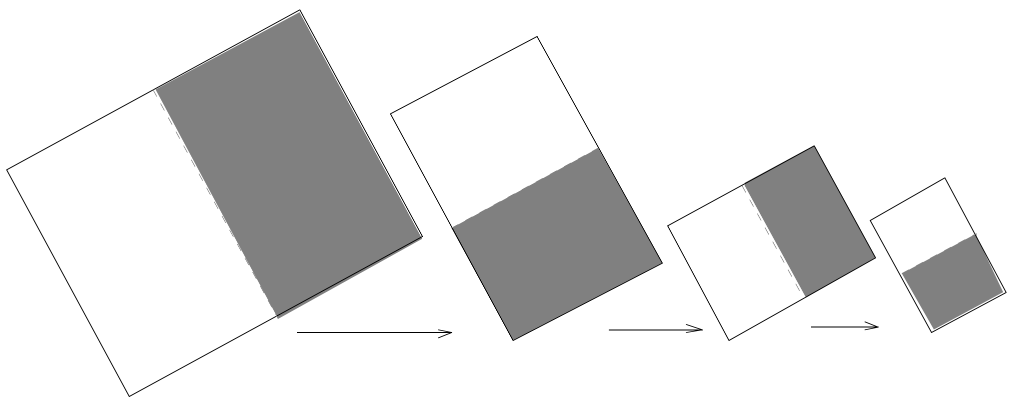


The rectangle after several tears have been made

Step 1: Take two standard $8\frac{1}{2}'' \times 11''$ sheets of paper (make sure they are both good rectangles). Set one aside for now; with the other, very carefully fold it in half along the longer side. Check that all the edges line up, and crease it thoroughly so that you can tear the sheet cleanly in half. Tear it in half to form two new rectangles.

Step 2: Take one of the half sheets; fold and tear it very cleanly in half (again along the longer side), saving the other half sheet for later comparison.

Step 3: Take the torn-off half sheet and tear it very cleanly in half, saving one half for later comparison.



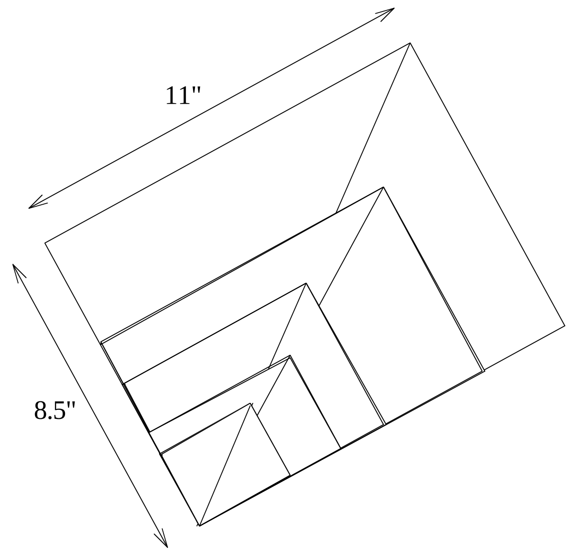
The shaded part gets folded and torn next.

You should be able to get to Step 4 or 5 before the pieces become too small to fold and tear.

1. You now have lots of rectangles—the rectangles you've torn, along with the first rectangle that's still whole. Hold each rectangle so that the longer side is along the bottom. Draw a diagonal on each rectangle from the bottom left corner to the upper right corner. Line up the edges of any two rectangles, making sure a pair of corners that include diagonals lies flush on top of each other. See if the two diagonals line up as well.

a. When the diagonals do line up, are the two rectangles scaled copies of each other?

b. When the diagonals do not line up, are the two rectangles scaled copies of each other?



Some of the diagonals might line up . . .

You'll prove this rule later.

2. Based on your findings, what seems to be a general rule for deciding when two rectangles are scaled copies of each other?

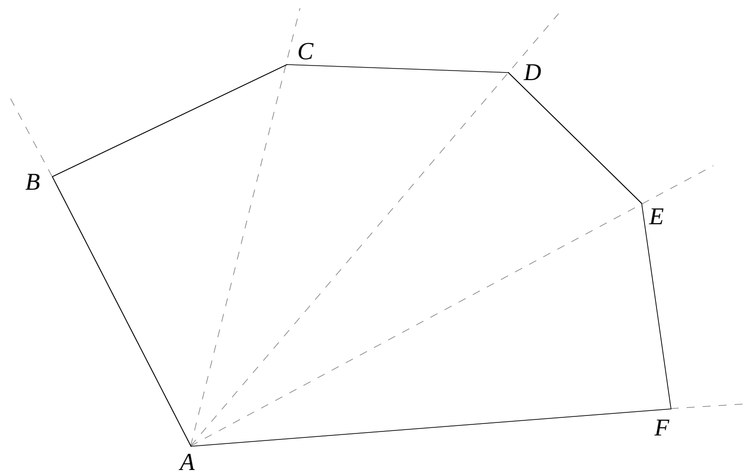
3. **Write and Reflect** Explain why this folding and tearing technique creates two sets of scaled rectangles.

POLYGON DIAGONALS

Draw a pentagon. Use a photocopy machine to make several reductions and enlargements of it, all at different sizes. Cut out your polygons, and stack them in order of size (largest on the bottom) so they're all oriented the same way and meet at a corresponding vertex.

4. How are the other corresponding vertices arranged?

5. Copy the figure below. Using a ruler, draw onto the figure a copy of *ABCDEF* that is scaled by a factor of $\frac{1}{2}$. How did you do it?

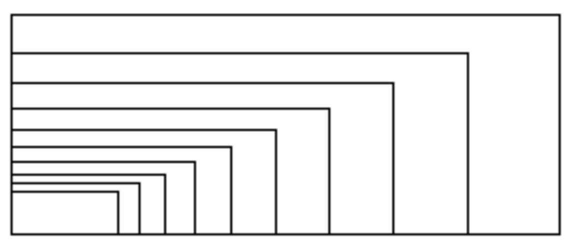


TAKE IT FURTHER

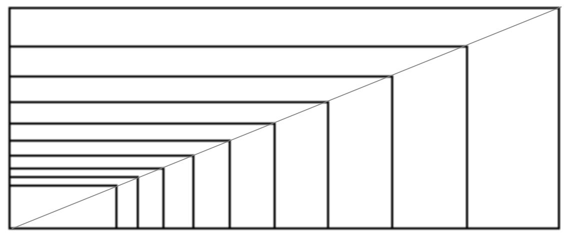
Here's a Logo procedure that draws rectangles:

```
to Rect :a :b
  repeat 2 [fd :a rt 90 fd :b rt 90]
end
```

Using this procedure, Suzanne draws the following "nest" of rectangles:



Suzanne is very proud of the fact that the corners of the rectangles "line up."



6. Use the Rect procedure to draw a nest of rectangles like Suzanne's.

7. Here's a procedure that makes nests of rectangles:

```
to Nest :a :b :n :k
 if :n = 0 [stop]
 rect :a :b
 nest :a*:k :b*:k  :n - 1 :k
end
```

To use it, try typing nest 20 50 10 1.5. Try other inputs. What does each input control in this procedure?

8. Here's another procedure that makes nests of rectangles:

```
to Nest1 :a :b :n :k
 if :n = 0 [stop]
 rect :a :b
 nest1 :a+:k :b+:k  :n - 1 :k
end
```

To use it, try typing nest 20 50 10 15. Try other inputs. What does each input control in this procedure?

9. Which of the above procedures always makes nests like Suzanne's? Explain.

Investigation 4.8

LIGHT AND SHADOWS: PROJECTED IMAGES

You have had lots of opportunities earlier in the module to examine the characteristics of scaled drawings. Now you will draw some yourself. In this section of the module, you will explore different techniques for making scale drawings and then prove why they work.

Some of the most common occurrences of scaled pictures are the images you see when you watch a slide show or a movie, the shadow you cast when walking in front of a bright flashlight, and the reductions and enlargements you make with a photocopy machine.

PROJECTORS

Here's an experiment you can try to help you understand how a movie projector works:

> With your arm stretched out in front of you, close one eye and hold your index finger and thumb just far enough apart to "bracket" the height of some object or person. Then bring your fingers closer to your eye, and see how you have to adjust them to make the object (or person) still fit just between them.

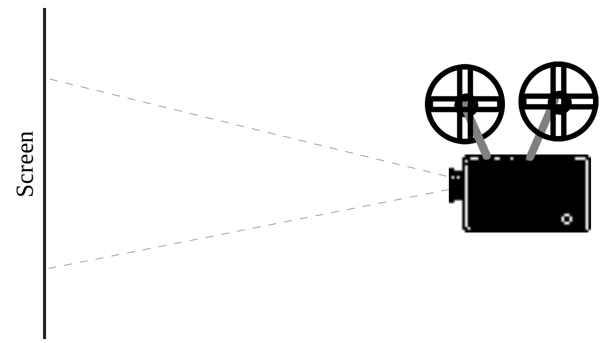
1. Make a sketch of the situation, with dots to show your eye, your fingers, and the ends of the image you are bracketing. Sketch a couple of "sight lines" that go from the eye, between the fingers, and to the object.

2. Describe what happens to the "bracket" distance between your fingers as you move them closer to your eye. Write an explanation of why it works this way.

A projector uses a very bright lightbulb and a series of lenses to make all the light coming from the projector act as if it is coming from one point. It is important that the rays of light not be parallel. The sun is far enough away for the light rays that reach the Earth to be nearly parallel. If you make shadows in the light of the sun, you will find it very hard to change the *size* of the shadow without also distorting its *shape*.

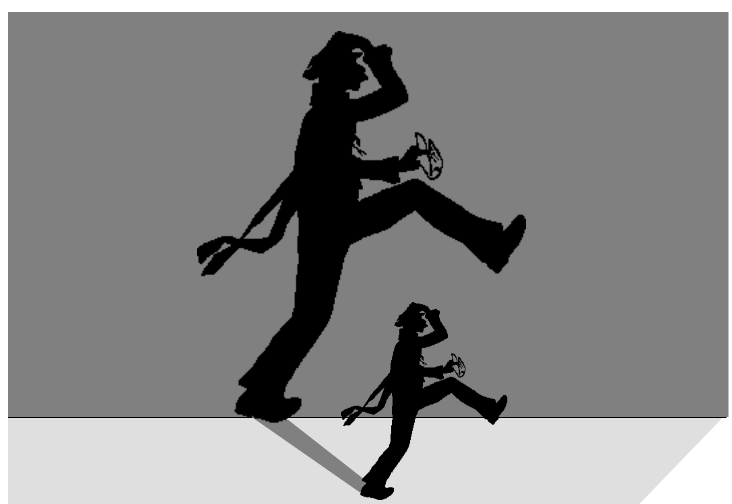
Since this is a side view of a movie projector, there is no way to get an idea of what image is being projected. Imagine some tiny picture on a frame of film getting enlarged by this process, and appearing on the screen.

Here is a sketch that shows why a projector can make a large image from a small original.



3. Describe what you could do to make the image on the screen smaller or larger. What might you do to make the image distorted?

SHADOWS



You can use a bright flashlight or a slide projector.

Find a light source that you can use to cast shadows. Play with shadow-casting; figure out what you have to do to make shadows larger and smaller. Find out how to make the shadow have the same shape as the original, and how to make the shadow distorted.

4. Write a brief set of rules that describes how to affect the shadows you make.

5. Describe and make a sketch to explain the following:

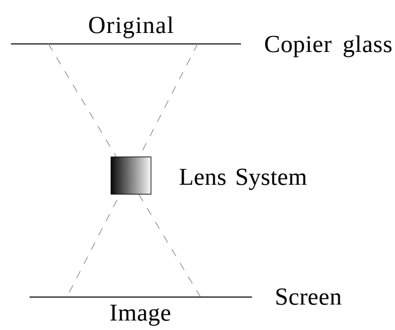
a. Where, in relation to the light source, would you put an $8\frac{1}{2}'' \times 11''$ piece of paper to cast a $17'' \times 22''$ shadow on a wall?

b. Where would you put the piece of paper to cast a same-size shadow?

TAKE IT FURTHER

Copy machines use principles similar to those of a projector, with a little more complexity. Copy machines can make both reductions and enlargements because the image is focused *through* a lens, and the distance from the lens to the "screen" (the part of the machine that picks up the image) can be varied. Here is a rough sketch:

The image is picked up by a light-sensitive drum or belt, then transferred and bonded to the piece of paper that comes out of the machine.



In some machines, if the distance from the original to the lens is the same as the distance from the lens to the image, the copy is made at 100%.

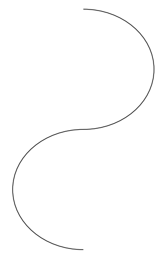
6. How do you think a copy is made larger or smaller? Make a sketch that shows an example of each situation.

The lens system of a real copy machine is actually more complicated than the description here, and uses mirrors to position, as well as to scale, the image.

Investigation 4.9

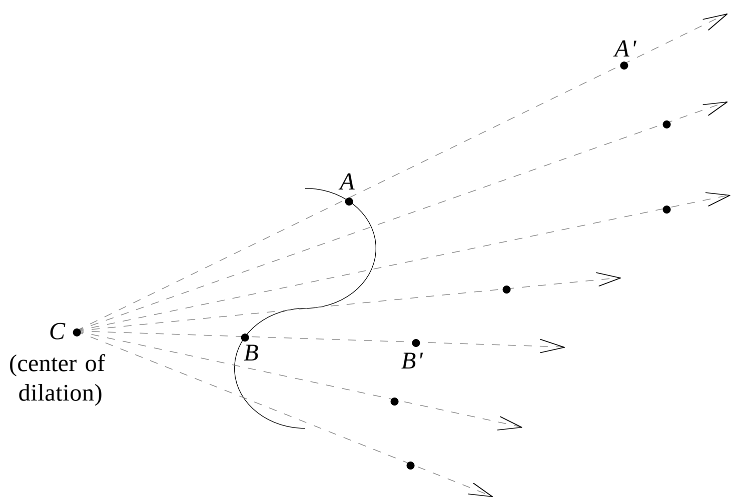
Curved or Straight? Just Dilate!

If all you had available was a ruler and pencil, how could you scale this curve by a factor of 2?



The discussion about movie projectors in Investigation 4.8 can help here. To scale a figure, a projector sends beams of light through it and catches those beams on a parallel surface. This is a "point-by-point" process. A two-dimensional model might look like this:

Since we're enlarging by a factor of 2, point A' is twice as far from the center of dilation as point A is. Concisely put, $CA' = 2CA$. Likewise, $CB' = 2CB$.



Dilating by a factor of 2

In mathematics, this process is called *dilation*. The point marked *center of dilation* represents the source of light, and the rays coming out from it represent the beams of light. The points farther out along the rays (like A' and B') represent some of the points on the enlarged copy of the curve. To get more points, draw more rays.

You can think of dilation as a particular way of scaling a figure. So, if you're asked to "dilate a figure by 2," this means to use the dilation method to scale it by 2.

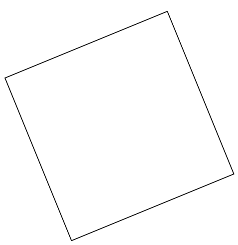
Of course, you can't dilate *every* point on a curve, because curves have infinitely many points. The more points you choose, though, the more accurate an enlargement or reduction you will get.

If you look in a dictionary, you will find that the word "dilate" means "to make wider or larger," or "to cause to expand." For instance, an eye doctor might dilate your pupils. In mathematics, the word is used more generally. Dilating can refer to either expanding a figure *or* shrinking it.

Pick any center of dilation that you want.

1. **a.** Draw a circle on a piece of paper and use the dilation method to scale it by a factor of 2. Choose enough points on the circle so that you can judge whether your dilation really does produce a scaled copy.

b. Compare your result with your classmates' work. How does your choice of the center of dilation affect the result?



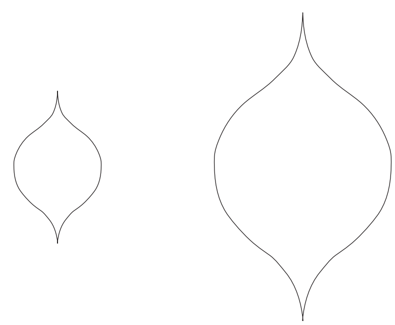
A tilted square

2. **a.** Draw a tilted square on a piece of paper and dilate it by a factor of 2 (dilate at least eight points before drawing the entire dilated square).

b. How does the orientation of the dilated square compare with the original tilted square?

3. Explain how to dilate a circle or square by a factor of $\frac{1}{2}$. Also explain how to dilate them by a factor of 3.

4. The picture below shows an ornamental pattern on the left, along with a larger copy on the right that was dilated by 2. Find the location of the center of dilation.

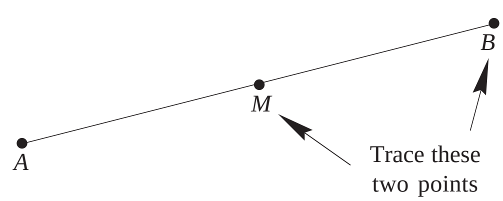


5. Make a scaled sketch of any picture you choose by dilating it by a factor of 2. One possibility is to use the picture of the head of Trig the horse shown below. You don't need to scale all the details from your picture (a rough outline is fine), but include at least the important ones.

Using Geometry Software to Dilate More Points

Dilating a curve with a ruler and pencil takes a lot of patience! You need to dilate a lot of points to get a good outline of the scaled copy. Geometry software gives a way to speed up this process.

6. Draw a segment $\overline{AB}$ with geometry software and construct its midpoint, M. Select points B and M, and use the software's "Trace" feature to indicate that you want the software to keep track of their paths.



Point *A* should stay fixed as you move point *B*.

a. Move point B around the screen, and use it to draw a picture or perhaps to sign your name. Compare point B's path to the path traced by point M. Are they the same? What is the relationship between them?

b. Use the software's segment tool to draw a polygon on your screen. Now move point B along its sides. Describe the path traced by point M.

Move point *B* fairly quickly around the screen in a random path. If you move it slowly, the screen fills up with lots of traced segments and the picture becomes hard to see.

c. In addition to tracing the points B and M, also trace the entire segment $\overline{AB}$ as you move point B. How does your final picture illustrate the concept of dilation?

MIRROR, MIRROR ON THE WALL

Here's an application of dilation with surprising results:

Stand in front of a mirror (perhaps one in the bathroom) and use some soap to trace your image. Include features like your eyes, nose, mouth, and chin.

7. Take a ruler and measure a few of the distances on your mirror picture—how far apart are your eyes, how long is your mouth, how far is it from your chin to the top of your head?

8. Compare the distances you've measured on the mirror to the actual measurements of your face. Are they the same?

9. How can the concept of dilation help to explain your results?

10. Stand in front of the mirror and have a friend trace the image of your face that she sees. Do you get the same picture as before? Why?

Investigation 4.10

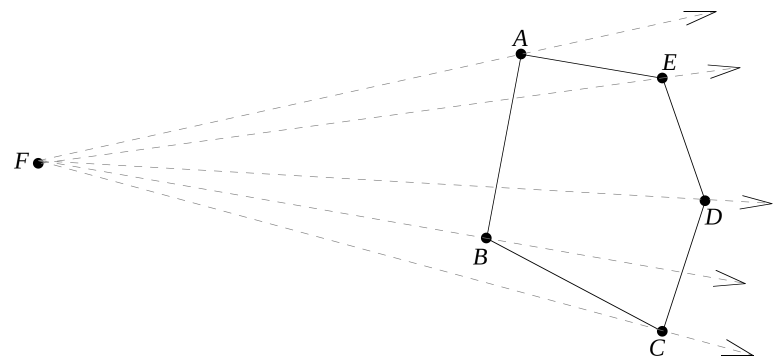
Ratio and Parallel Methods

When you scale a curve using dilation, you need to dilate a fair number of points before the scaled copy begins to even look recognizable. For simple figures like polygons, though, there are two shortcuts—the ratio and parallel methods—that make the dilation process much faster.

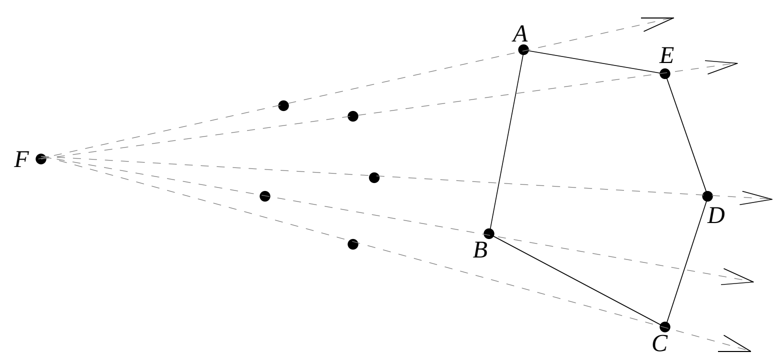
The Ratio Method for Dilating a Polygon

Suppose that you want to dilate a polygon like *ABCDE* by $\frac{1}{2}$. Draw your own polygon and pick any center of dilation (point *F* below). Draw rays from that point through every vertex of the polygon:

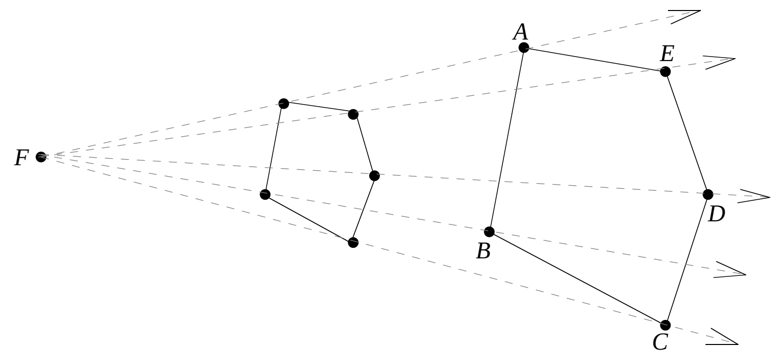
Try this with your own polygon.



Next, find the midpoints of segments $\overline{FA}$ through $\overline{FE}$:



Finally, connect the midpoints to form a new polygon:



Take enough measurements (sides and angles, say) to convince yourself.

1. Is your new polygon a scaled (by $\frac{1}{2}$) copy of the original? How can you tell?

2. There seem to be many parallel segments in the figure above. Label the parallel segments you see on your own drawing.

3. Does the ratio method work if point F is inside the polygon?

4. Does it work if point F is *on* $ABCDE$? Try placing F at a vertex of the polygon.

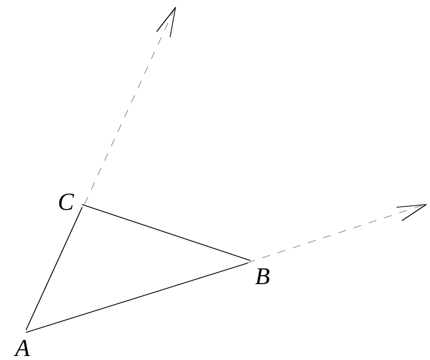
Aside from making half-size reductions, the ratio method can dilate a figure by any amount you choose.

5. Start with a polygon and use the ratio method to dilate it by $\frac{1}{3}$.

6. Use the ratio method to enlarge a polygon by a factor of 2.

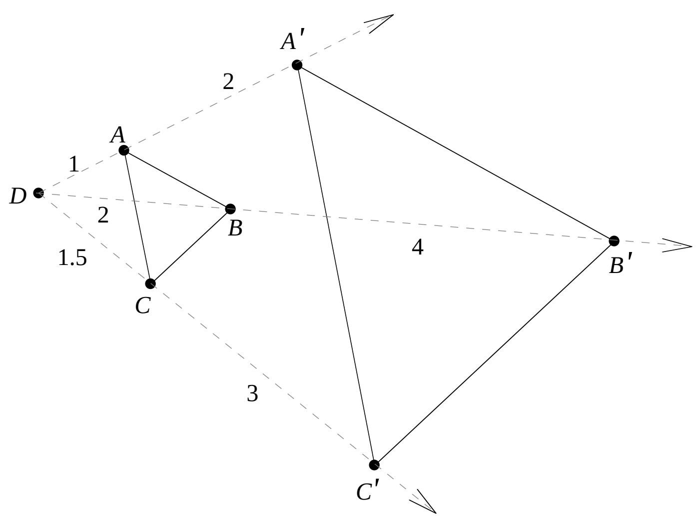
7. Rosie wants to make two scale drawings of $\triangle ABC$: one that is dilated by a factor of 2, and another that is dilated by a factor of 3. She decides to make the triangle

vertex A her center of dilation. Draw a picture like the one below and finish her construction.



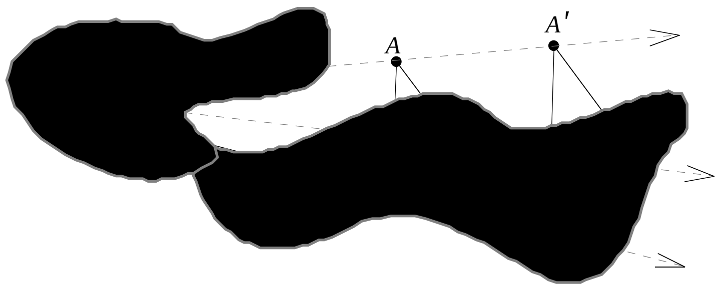
Steve uses the ratio method to enlarge $\triangle ABC$ below by a factor of 2. He follows this procedure:

- He measures the distance DA and finds it to be 1.0. So he moves out along the ray DA until he finds a point A' so that $AA' = 2$ (twice as long as DA).
- He measures the distance DB and finds it to be 2.0. So he moves out along the ray DB until he finds a point B' so that $BB' = 4$ (twice as long as DB).
- He measures the distance DC and finds it to be 1.5. So he moves out along the ray DC until he finds a point C' so that $CC' = 3$ (twice as long as DC).
- He draws $\triangle A'B'C'$.



8. To Steve's surprise, $\triangle A'B'C'$ has sides that are proportional to $\triangle ABC$ but are not twice as long. How many times as long are they? What's the matter here?

9. Oh, no! There's been a problem at the printing press. The picture below was supposed to show $\triangle ABC$ and its dilated companion, $\triangle A'B'C'$, but an ink spot spilled onto the page. Can you salvage this disaster by calculating by how much $\triangle ABC$ has been scaled?



THE PARALLEL METHOD FOR DILATING A POLYGON

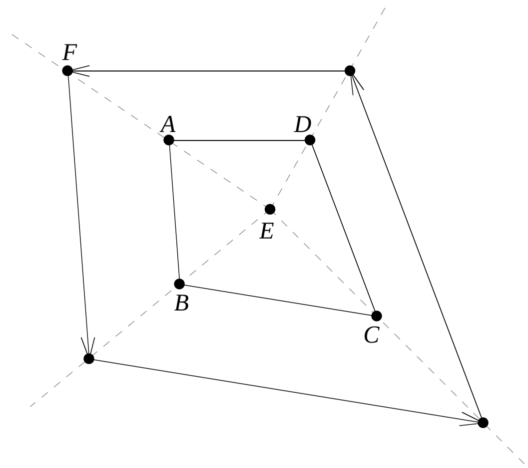
Take a look back at each picture you've drawn of a polygon and its dilated companion. You might notice something — the polygons are oriented the same way and their sides are parallel. Aha! This observation suggests another way to dilate.

The *parallel method* begins with the same setup as the ratio method:

- To make a drawing of quadrilateral $ABCD$ that's dilated by a factor of 2, pick any point E (the center of dilation) and draw rays from that point through all the vertices.
- Go out along one ray ($\overrightarrow{EA}$ on the next page) twice the distance from E to the polygon vertex. Mark this location (point F on the next page).

Estimate the parallel segments as best you can.

- Starting at point F, move around the rays, drawing segments parallel to the sides of polygon $ABCD$.



Start drawing parallels to the sides of ABCD from point F.

Point F is chosen so that $EF = 2EA$.

10. Is the new polygon a scaled (by a factor of 2) copy of the original? How can you tell?

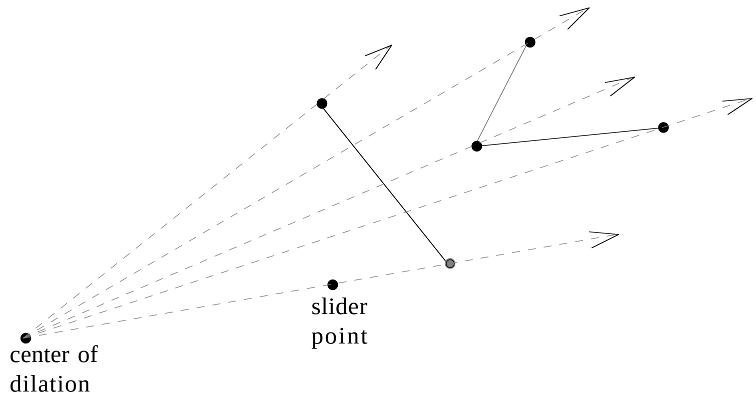
11. Use this technique to reduce a polygon so that it's dilated by $\frac{1}{2}$. Also, enlarge a polygon by a factor of 3. Be sure to try locations for the center of dilation that are inside the polygon, on the polygon, and outside the polygon.

USING A SLIDER POINT

If you try the parallel method with geometry software, you can create a whole series of polygons dilated to different sizes without having to start from scratch each time. Here's how:

Draw a polygon with the software, pick a center of dilation, and draw rays from this point through the polygon's vertices. Then place a point anywhere along one of the

rays. This will be your "slider point," and it will control the amount of dilation.

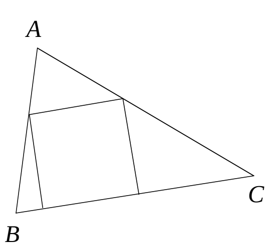


With the slider point as your place to begin, use the parallel method to make a dilated copy of the polygon. When your dilated polygon is complete, move the slider point back and forth along its ray. The polygon will grow and shrink, always remaining a scaled copy of the original!

12. Use the software to calculate by how much your polygon has been dilated. This dilation amount should update itself automatically as you change the location of the slider point.

- **a.** For what locations of the slider point is the dilation amount less than one?
- **b.** For what locations of the slider point is the dilation amount greater than one?
- **c.** Where is the dilation amount equal to one?

TAKE IT FURTHER. . . . .



13. Draw a triangle ABC with pencil and paper or geometry software. Your challenge is to construct a square with one side lying on $\overline{BC}$ and the other two vertices on sides $\overline{AB}$ and $\overline{AC}$. One way to get started is to draw a square with side $\overline{BC}$ (facing away from the triangle) and think of A as a dilation point.

Investigation 4.11

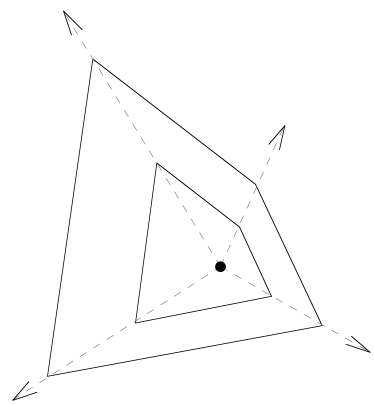
Nested Triangles: Building Dilated Polygons

Now that you've experimented with the ratio and parallel methods for dilating polygons, the question remains: Why do these methods make scaled copies?

Since polygons can have 3, 4, 5, or even more sides, finding a general proof that works for *any* polygon with *any* number of sides can be tricky. But all polygons can be subdivided into triangles. If we can show that the ratio and parallel methods work for triangles, then perhaps there's an easy way to extend the results to all polygons.

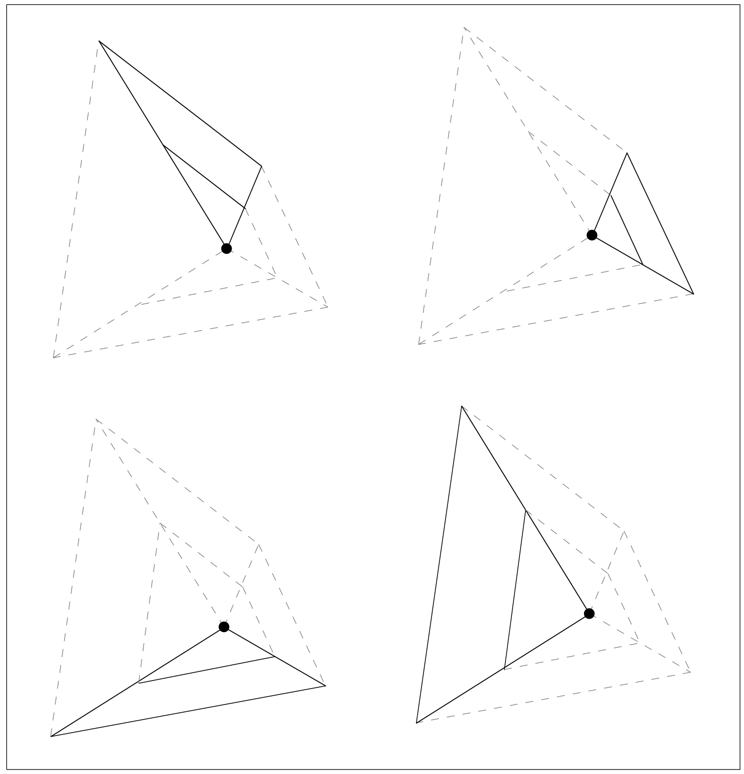
From Polygons to Triangles

Here is a polygon and a dilated copy of it that was made using the parallel method:



A polygon and its scaled companion

The four pictures on the next page show the same polygons, but now, a pair of *nested triangles*—one triangle sitting inside another—is highlighted in each. Notice that each pair of nested triangles contains a side from the original polygon and a parallel side from the scaled polygon.



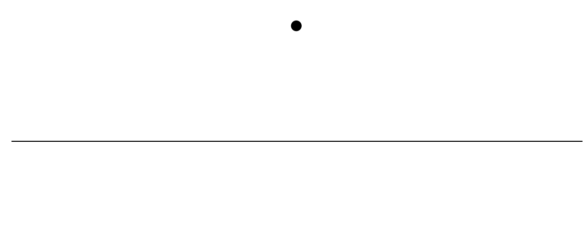
Four pairs of nested triangles

THREE TRIANGLE EXPERIMENTS

Here are three experiments to start you thinking about the properties of nested triangles.

EXPERIMENT ONE

Draw a picture like the one below that contains two parallel lines and a point, where the distance between the point and the top line is equal to the distance between the two lines.

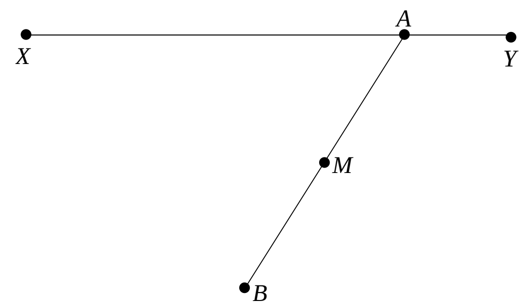


1. Using your picture, find a super-fast way to take a ruler and draw ten segments that will each have their midpoints automatically marked.

2. Now, find a really quick way to draw ten segments that will each be divided in the ratio of 1 to 3. You will need to reposition the two parallel lines and the point to do this.

EXPERIMENT TWO

Use geometry software to draw a segment, $\overline{XY}$. Then construct a point A on the segment and a point B that's not on $\overline{XY}$. Draw $\overline{AB}$ and construct its midpoint, M. Drag point A back and forth along the entire length of $\overline{XY}$ while tracing the path of point M.



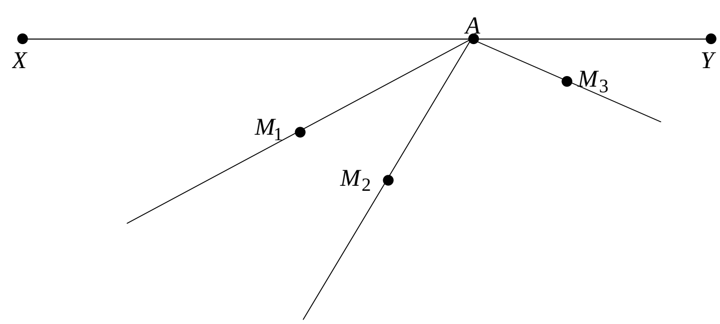
Drag point A along $\overline{XY}$.

3. a. Describe the path traced by M.

b. How do the length and orientation of the path traced by point M compare to the length and orientation of $\overline{XY}$?

4. Repeat Experiment Two, only this time, instead of constructing the midpoint of $\overline{AB}$, place the point M somewhere else on the segment. How does the position of M affect your answers to Problem 3?

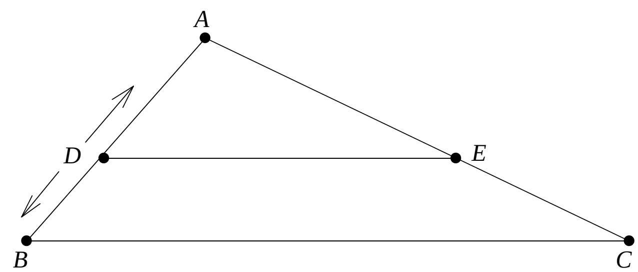
5. Using geometry software, draw a segment, $\overline{XY}$, and construct a point A on the segment. Then draw three segments, each with A as the endpoint. Construct the midpoints M_1, M_2, and M_3 of the three segments. Move point A back and forth along $\overline{XY}$ while tracing these midpoints. Describe the paths of the midpoints and anything you can say about their lengths.



EXPERIMENT THREE

Use geometry software to draw an arbitrary triangle ABC. Place a point D anywhere on side $\overline{AB}$ and then construct a segment $\overline{DE}$ that is parallel to $\overline{BC}$.

$\triangle$ *ADE* and $\triangle$ *ABC* are a pair of nested triangles.



Drag point D along $\overline{AB}$.

6. **a.** Use the software to calculate the ratio $\frac{AD}{AB}$.

b. Find two other ratios of lengths with the same value. Do all three ratios remain equal to each other when you drag point D along $\overline{AB}$?

c. Use the software to calculate the ratio $\frac{AD}{DB}$. Find any other ratios sharing this same value.

You can also think about this setup as a dilation. The center of the dilation is A, and the scale factor is $\frac{AB}{AD}$. We say that $\overline{DE}$ splits $\overline{AB}$ and $\overline{AC}$ proportionally.

DEFINITION

Let** ABC **be a triangle with** D **a point on** $\overline{AB}$ **and** E **a point on** $\overline{AC}$**.** $\overline{DE}$ **splits** $\overline{AB}$ **and** $\overline{AC}$ **proportionally if

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

The scale factor $\frac{AB}{AD}$ is called the ***common ratio***.

FOR DISCUSSION

Use the results from these three experiments to propose some possible theorems about nested triangles. Compare the conjectures you make with those of your classmates. Which do you think are true?

TWO THEOREMS ABOUT NESTED TRIANGLES

The following two theorems may have been suggested by you or your classmates. You will prove them in the next investigation.

THEOREM 4.1 *The Parallel Theorem*

If a segment is parallel to one side of a triangle, then

1. it splits the other two sides proportionally, and
2. the ratio of the length of the parallel side to this segment is equal to the common ratio.

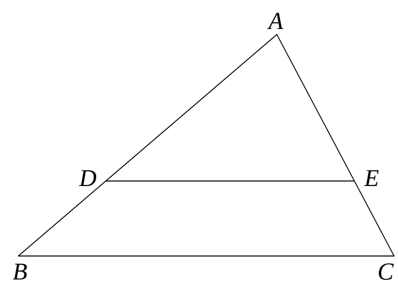
THEOREM 4.2 *The Side-Splitting Theorem*

Try not to laugh too hard.

If a segment splits two sides of a triangle proportionally, then it is parallel to the third side.

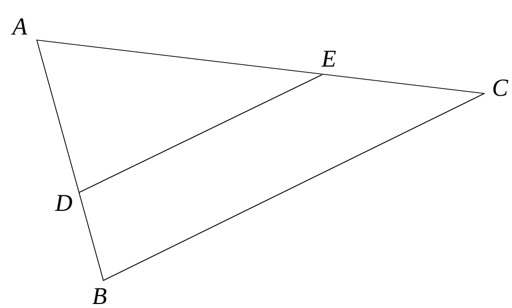
7. Understanding these theorems by reading through the words can be pretty tricky. Why don't we replace some of the words with actual segment names? Rewrite the two theorems using the nested triangles that follow as a reference. Make each

theorem as specific as possible—if a segment length or proportion is mentioned, substitute that segment or proportion in place of the words.



8. The Parallel Theorem says that a segment parallel to a side of a triangle "splits the other two sides proportionally." Tammy Jo has two ways to remember this. She says, "part is to part as part is to part" or "part is to whole as part is to whole." What does she mean?

9. In the picture below, $\overline{DE} \parallel \overline{BC}$:



The picture is not drawn to scale.

a. If $AD = 1$, $AB = 3$, and $AE = 2$, what is AC?

b. If $AE = 4$, $AC = 5$, and $AB = 20$, what is AD?

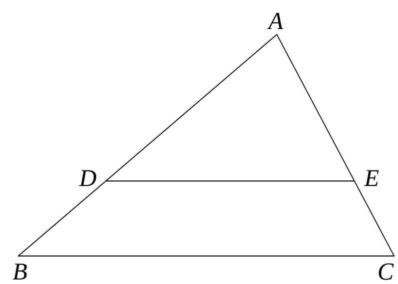
c. If $AD = 3$, $DB = 2$, and $AE = 12$, what is EC?

d. If $AE = 1$, $AC = 4$, and $DE = 3$, what is BC?

e. If $AD = 2$ and $DB = 6$, what is the value of $\frac{DE}{BC}$?

TAKE IT FURTHER

By now, you've seen that there are two ways to write the first part of the Parallel Theorem. If $\overline{DE} \parallel \overline{BC}$ in the figure below, then $\frac{AB}{AD} = \frac{AC}{AE}$ *and* $\frac{DB}{AD} = \frac{EC}{AE}$.



The next two problems show you how to prove that these proportions are two different ways of writing the same information.

Hint: $\frac{r-s}{s} = \frac{r}{s} - \frac{s}{s}$.

10. First, let's prove a related fact using algebra. Suppose that r, s, t, and u are any four numbers. If $\frac{r}{s} = \frac{t}{u}$, explain why it's also true that $\frac{r-s}{s} = \frac{t-u}{u}$.

11. Using the previous problem as a guide, explain why the proportion $\frac{AB}{AD} = \frac{AC}{AE}$ is equivalent to $\frac{DB}{AD} = \frac{EC}{AE}$.

12. Write and Reflect The Midline Theorem says that a segment connecting the midpoints of two sides of a triangle is parallel to the third side and half as long. Draw a picture for the statement of this theorem. This is just a special case of either the Parallel or Side-Splitting Theorem. Which one? Explain how it connects.

SIDE-SPLITTING AND PARALLEL THEOREMS

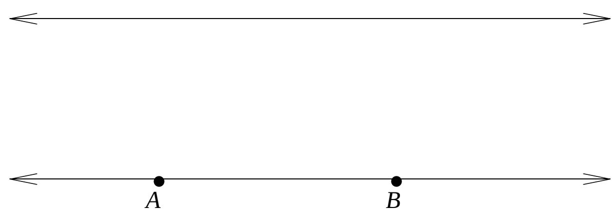
PROVING THE SIDE-SPLITTING AND PARALLEL THEOREMS

In this investigation, you will construct proofs for the Side-Splitting Theorem and the two parts of the Parallel Theorem. At the core of the proofs is an ingenious area argument devised by Euclid. To prepare for the proofs, here are several questions about area to work on first. You will use these results in the proofs that follow.

COMPARING AREAS: SOME WARM-UPS

Geometry software makes it easy for you to experiment with different locations of point *C*.

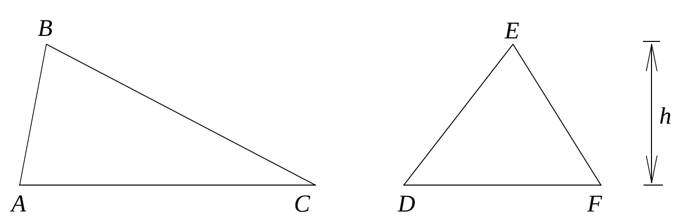
1. Points A and B of $\triangle ABC$ are fixed on the line below. Point C, which is not shown, lies on the parallel line above it and is free to wander anywhere along it. For what location of point C is the area of $\triangle ABC$ the largest? Why?



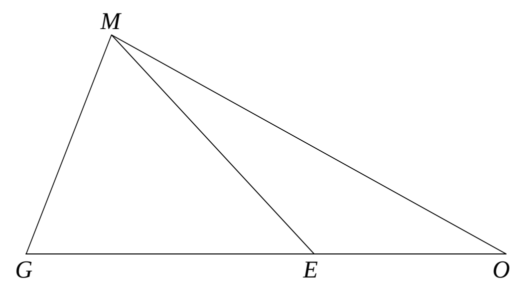
2. This problem asks you to prove the following statement:

> *If two triangles have the same height, the ratio of their areas is the same as the ratio of their bases.*

a. Both $\triangle ABC$ and $\triangle DEF$ have the same height, h. Write an expression for the area of each triangle, and then show that the ratio of their areas is equal to $\frac{AC}{DF}$.



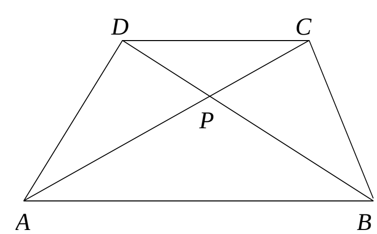
b. The area of $\triangle GEM$ is 3 square inches and the area of $\triangle MEO$ is 2 square inches. What is the value of $\frac{GE}{EO}$? Why? Also find the value of $\frac{GE}{GO}$.



3. $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{DC}$.

a. Explain why the area of $\triangle ACB$ is equal to the area of $\triangle ADB$.

b. Name two other pairs of triangles in the figure that also have equal areas.



THE PARALLEL THEOREM

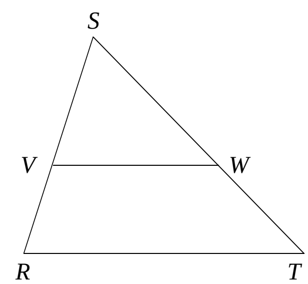
Recall what the first part of the Parallel Theorem says:

If a segment is parallel to one side of a triangle, then it splits the other two sides proportionally.

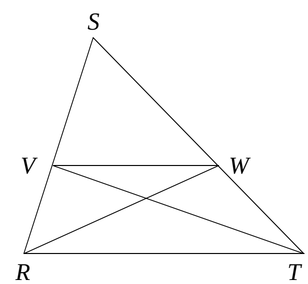
Here's how Euclid proved it:

In the figure below, $\overline{VW} \parallel \overline{RT}$. Show that

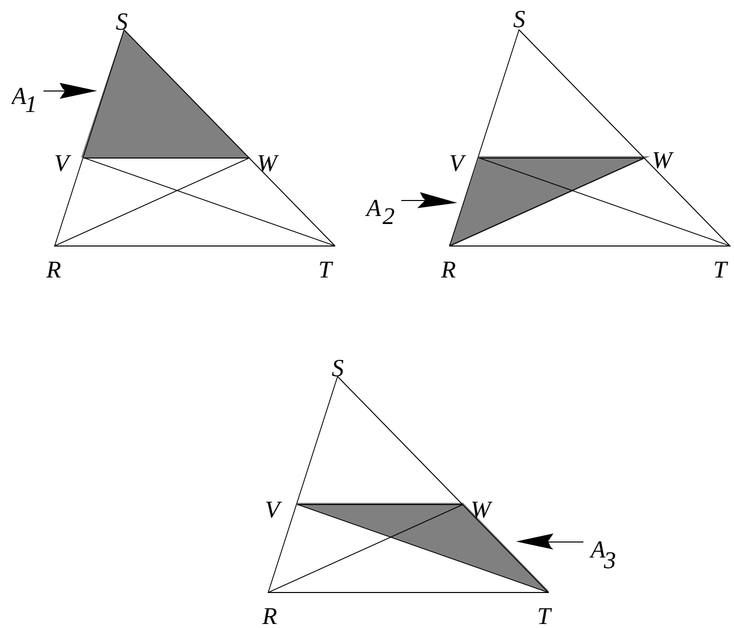
$$\frac{SV}{VR} = \frac{SW}{WT}.$$



Draw segments $\overline{RW}$ and $\overline{TV}$.



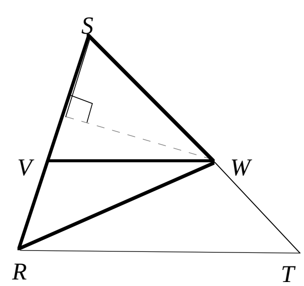
Let the area of $\triangle SVW$ be A_1, the area of $\triangle RVW$ be A_2, and the area of $\triangle TVW$ be A_3:



And now:

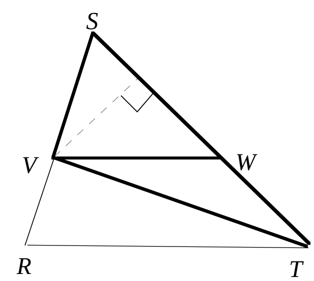
- Look sideways at $\triangle SVW$ and $\triangle RVW$. They have the same height (it's the perpendicular from W to $\overline{RS}$), so their bases $\overline{SV}$ and $\overline{VR}$ have the same ratio as their areas:

$$\frac{A_1}{A_2} = \frac{SV}{VR}.$$



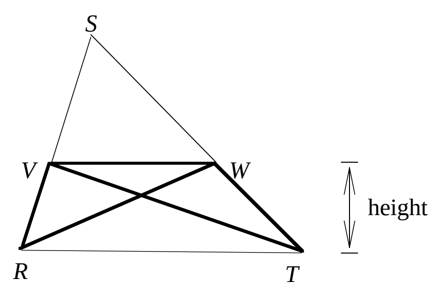
- Look sideways at $\triangle SVW$ and $\triangle TVW$. They have the same height (it's the perpendicular from V to $\overline{TS}$), so their bases $\overline{SW}$ and $\overline{WT}$ have the same ratio as their areas:

$$\frac{A_1}{A_3} = \frac{SW}{WT}.$$



- Look at $\triangle RVW$ and $\triangle TVW$. They have the same base ($\overline{VW}$) and the same height (it's the distance between the parallels $\overline{VW}$ and $\overline{RT}$), so they have the same area:

$$A_2 = A_3.$$



Since $A_2 = A_3$,

$$\frac{A_1}{A_2} = \frac{A_1}{A_3}.$$

Thus,

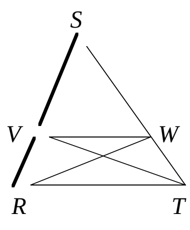
$$\frac{SV}{VR} = \frac{SW}{WT},$$

so the proof is complete.

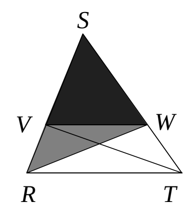
A QUICK OVERVIEW

Once you've had the chance to digest all the nitty-gritty details of Euclid's proof, it's nice to step back to think about its overall structure. Here's the same proof as above, only condensed. All of the algebraic steps and extra details are omitted to leave just the key concepts:

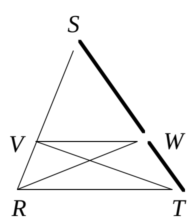
We need to show that $\frac{SV}{VR} = \frac{SW}{WT}$. Since $\overline{SV}$ and $\overline{VR}$ belong to triangles sharing a common height, we can say:



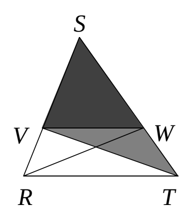
$$\frac{SV}{VR} = \frac{\text{area }(\Delta\ SVW)}{\text{area }(\Delta\ RVW)}$$



We can write the same kind of proportion for $\overline{SW}$ and $\overline{WT}$ since they also belong to triangles sharing a common height:



$$\frac{SW}{WT} = \frac{\text{area }(\Delta\ SVW)}{\text{area }(\Delta\ TVW)}$$



Both numerators of the area ratios are the same—they're the area of $\triangle SVW$. The denominators are equal too because both $\triangle RVW$ and $\triangle TVW$ share a common base and have the same height. So the area ratios are equal. Thus, $\frac{SV}{VR} = \frac{SW}{WT}$.

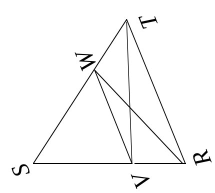
WAYS TO THINK ABOUT IT

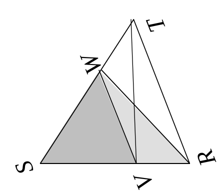
Start with what you know and look for inspiration.

How in the world did Euclid come up with his proof? In particular, what made him think to look at area? Well, we don't know for sure, but here's how a student might reason about it:

> "I'm trying to show that $\frac{SV}{VR} = \frac{SW}{WT}$. Somehow I need to use the given hypothesis that $\overline{VW} \parallel \overline{RT}$. Offhand, I don't see any connection between that hypothesis and the conclusion.
>
> How about those triangles sitting inside the big triangle, $\triangle SRT$? There's $\triangle SVW$, $\triangle RVW$, $\triangle TVW$, and some others. Do I know anything about them?
>
> Well, I'm trying to prove that $\frac{SV}{VR} = \frac{SW}{WT}$. The lengths of two of the sides of $\triangle SVW$ appear in that proportion—namely, $\overline{SV}$ and $\overline{SW}$. What does that leave? Well, $\overline{VR}$ is a part of $\triangle RVW$ and $\overline{WT}$ is a part of $\triangle TVW$. Those two triangles share some things in common. They both have a base of $\overline{VW}$ and they both have the same height. Hmmm . . . same base and same height . . . that means they have the same area. I wonder if area can help me?

Thinking to look at area is the inspiration needed to prove this theorem. Inspirations are the hardest thing to teach in mathematics; there is no general method that guarantees a useful insight.



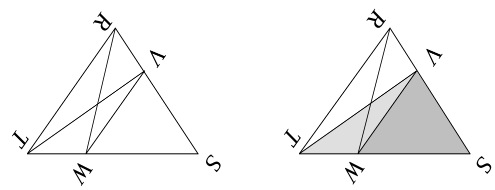


> If I turn the whole picture on its side and look at $\triangle SVW$ and $\triangle RVW$, it reminds me of a problem I've seen before (Problem 2b). That one had to do with area. Both $\triangle SVW$ and $\triangle RVW$ go up to point W, so they have the same height. That means the ratio of their areas is like the ratio

of their bases. I'll write that down and see if it helps:

$$\frac{\text{area of } \triangle SVW}{\text{area of } \triangle RVW} = \frac{SV}{VR}.$$

Now I think I'm getting somewhere — that gives me the first part of the conclusion. Maybe if I turn the picture the other way, I can get the second part:



When it's over on this side, I can see that $\triangle SVW$ and $\triangle TVW$ have the same height. So I can set up a ratio again using their bases. If I write it out, I get

$$\frac{\text{area of } \triangle SVW}{\text{area of } \triangle TVW} = \frac{SW}{WT}.$$

That's the second part of the conclusion. I hope I can find a connection between the two parts. Both of the ratios I came up with use the area of $\triangle SVW$. And the areas for the other two triangles, $\triangle RVW$ and $\triangle TVW$, are the same! I noticed that earlier. Now let me write this out:

$$\frac{\text{area of } \triangle SVW}{\text{area of } \triangle RVW} = \frac{\text{area of } \triangle SVW}{\text{area of } \triangle TVW}$$

means that

$$\frac{SV}{VR} = \frac{SW}{WT}.$$

Pretty good—Euclid's got nothing on me!"

4. **Write and Reflect** How were *you* thinking when you worked through the proof of the Parallel Theorem?

THE SIDE-SPLITTING THEOREM

And now for the Side-Splitting Theorem:

If a segment splits two sides of a triangle proportionally, then it is parallel to the third side.

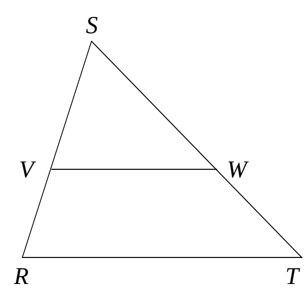
5. This time, the proof is up to you! Using the same setup as Euclid's proof, see if you can prove the theorem yourself. Then write up your proof so that somebody else can follow it.

THE PARALLEL THEOREM CONTINUED

Now we prove part b of the Parallel Theorem. Recall what this theorem says:

If $\overline{VW}$ is parallel to $\overline{RT}$, then

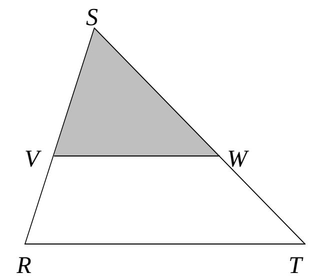
$$\frac{RT}{VW} = \frac{SR}{SV} = \frac{ST}{SW}.$$



You've proved part a of this theorem already—the Parallel Theorem guarantees that if $\overline{VW} \parallel \overline{RT}$, then $\frac{SR}{SV} = \frac{ST}{SW}$. (Why is this the same as saying $\frac{SV}{VR} = \frac{SW}{WT}$?) So all that remains is to prove their equality to $\frac{RT}{VW}$.

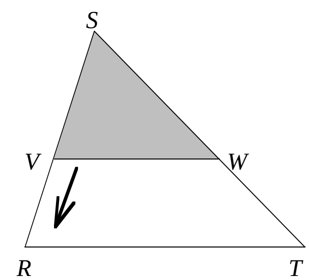
Try this with paper and scissors.

Imagine that $\triangle SVW$ and $\triangle SRT$ are separate paper triangles that you've cut out and placed on top of each other like so:



You take $\triangle SVW$ and slide it along $\overline{SR}$ until vertices V and R coincide:

Why will $\angle SVW$ fall right onto $\angle SRT$?



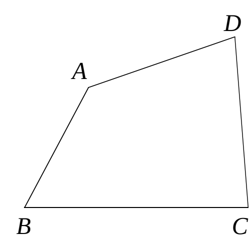
6. Draw a picture of what the two triangles will look like after $\triangle SVW$ has been slid all the way to $\overline{RT}$.

7. Which two segments are now parallel? Why?

8. Use this new setup to prove the remainder of the Parallel Theorem—namely, that $\frac{RT}{VW}$ equals both $\frac{SR}{SV}$ and $\frac{ST}{SW}$.

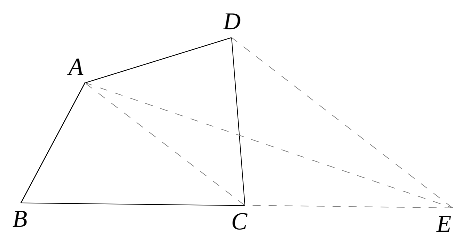
TAKE IT FURTHER

9. How can you construct (*without* taking any measurements) a triangle with the same area as quadrilateral *ABCD*?



Here's one method:

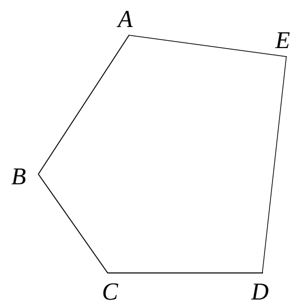
First, draw diagonal $\overline{AC}$. Then draw a line through point D parallel to $\overline{AC}$. Extend $\overline{BC}$ to meet the parallel at point E.



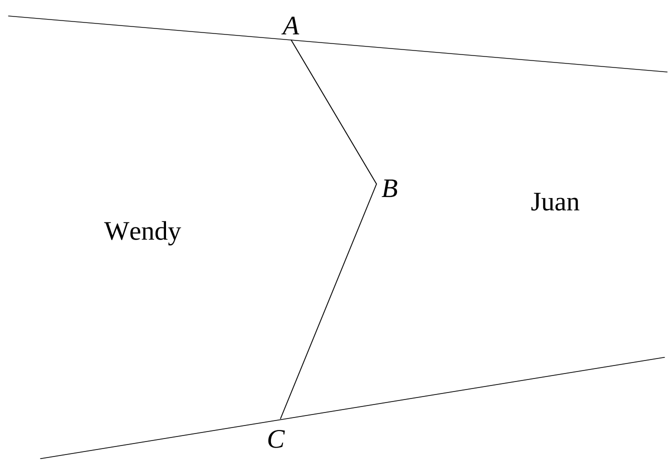
This completes the construction. The area of $\triangle ABE$ is equal to the area of the quadrilateral *ABCD*. Explain why.

Hint: Find another triangle that has the same area as $\triangle ADC$.

10. Extend the technique used in Problem 9 to construct a triangle with area equal to that of pentagon *ABCDE* below.



11. The picture below shows the land owned by Wendy and Juan. The border between their properties is represented by the segments $\overline{AB}$ and $\overline{BC}$. How can you replace these segments by just a single segment that does not change the amount of land owned by either person?



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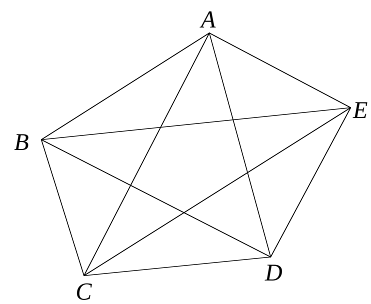
12. In pentagon *ABCDE* below, four of the sides are each parallel to a diagonal: $\overline{AB} \parallel \overline{CE}$, $\overline{BC} \parallel \overline{AD}$, $\overline{CD} \parallel \overline{BE}$, and $\overline{DE} \parallel \overline{CA}$. Prove that the remaining side $\overline{EA}$ is parallel to the diagonal $\overline{DB}$.



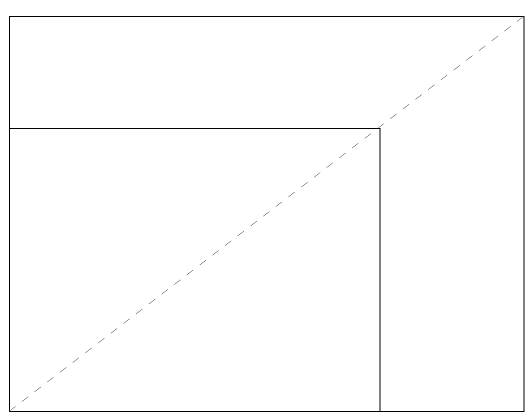
Here's an idea to get you started:

Proving that $\overline{EA} \parallel \overline{DB}$ is equivalent to showing that the area of $\triangle ABE$ is equal to the area of $\triangle ADE$. Why?

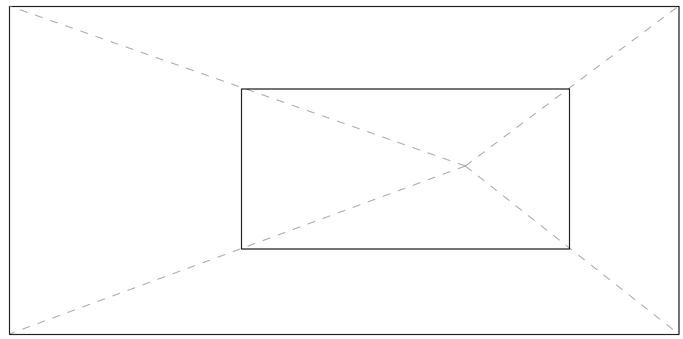
USING THE PARALLEL AND SIDE-SPLITTING THEOREMS

13. Now that you have proved the Parallel and Side-Splitting Theorems, use them to explain why the two rectangles in each figure below and on the next page are scaled copies of each other:

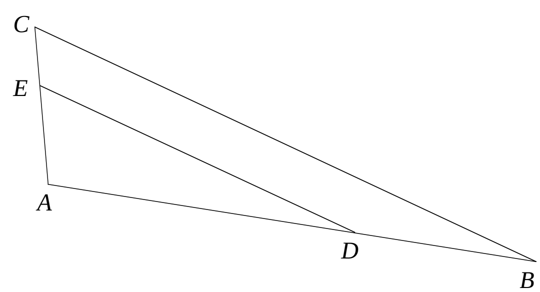
a.



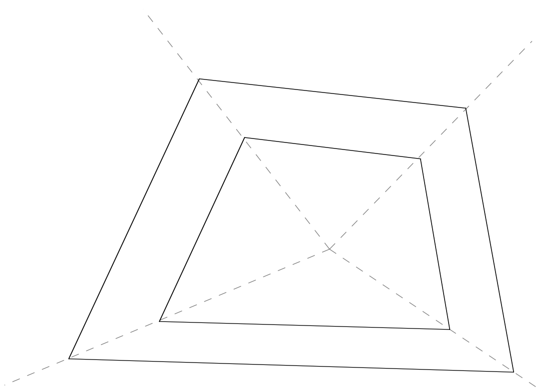
b. The sides of the inner rectangle are drawn parallel to the outer sides.



14. In the figure below, $\overline{DE} \parallel \overline{BC}$. Explain why $\triangle ABC$ is a scaled copy of $\triangle ADE$.



15. The sides of the outer polygon below are drawn parallel to the sides of the inner one. Explain why the polygons are scaled copies.

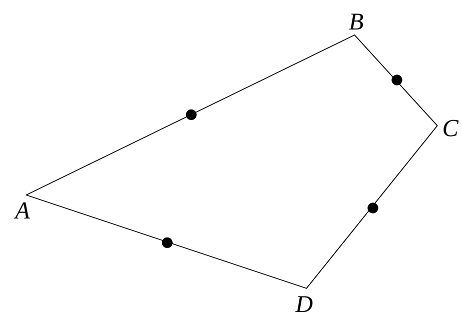


MIDPOINTS IN QUADRILATERALS

The Side-Splitting Theorem can help you prove a surprising result about quadrilaterals drawn within other quadrilaterals. Try this:

Draw any quadrilateral *ABCD* and construct the midpoint of each side. Then connect the midpoints to form a new quadrilateral inside of *ABCD*, which we will call a "midpoint quadrilateral."

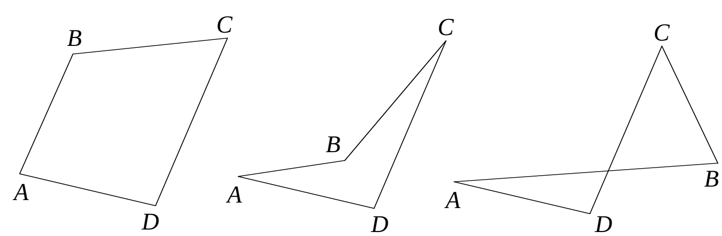
16. Describe the features of your new quadrilateral. Try to classify it as a particular kind of quadrilateral. If you're using geometry software, experiment by moving the vertices and sides of *ABCD*. Does the midpoint quadrilateral retain its special features?



A quadrilateral and its midpoints

Technically, self-crossing figures like the one on the right are not quadrilaterals, but your proof could work for these figures as well.

17. Prove that your conjectures about midpoint quadrilaterals are correct. Make sure that your proof is valid for each of the three locations of B shown below.



Three different locations of point B

18. The diagonals of a quadrilateral *ABCD* measure 8 inches and 12 inches. What is the perimeter of the midpoint quadrilateral?

19. Describe any special characteristics of the midpoint quadrilateral if the diagonals of the outer quadrilateral are

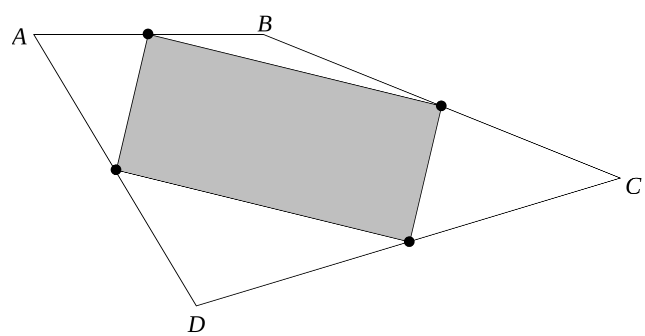
a. congruent;

b. perpendicular to each other;

c. congruent and perpendicular to each other.

TAKE IT FURTHER.................................

20. When the midpoints of any quadrilateral *ABCD* are connected, the inner quadrilateral (shaded below) is a parallelogram. Show that the area of the parallelogram is half the area of *ABCD*.

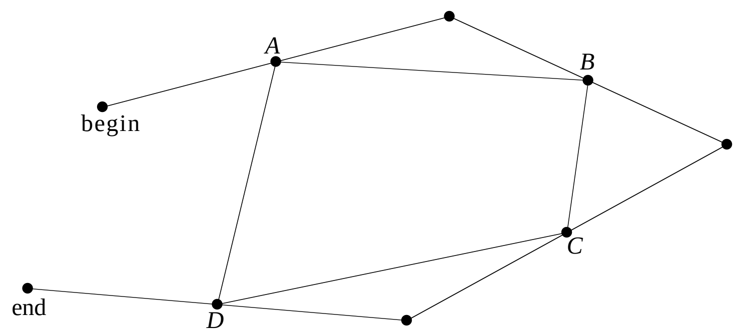


Hint: Can you arrange the four triangles sitting on the sides of the parallelogram to fit exactly within it?

21. Here's a nice problem to explore with geometry software:

Draw an arbitrary quadrilateral *ABCD* and place a point (labeled "begin" in the following figure) anywhere you like. Construct a segment through *A* with "begin" as an endpoint and *A* as its midpoint. Now start at the endpoint of this new segment and construct another segment, this time with *B* as the midpoint.

Continue doing this until you construct a segment with D as the midpoint. The finishing point of this whole journey around $ABCD$ is labeled "end."



a. Draw the segment that connects "begin" to "end." Drag the "begin" point around the screen and watch what happens to this segment. How does its length change? How does its slope change? Make some conjectures and see if you can prove them.

b. For what kinds of quadrilaterals $ABCD$ are "begin" and "end" exactly the same point?

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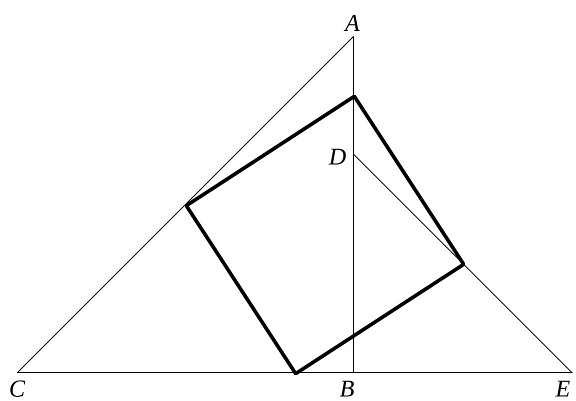
22. Take any two isosceles right triangles like $\triangle ABC$ and $\triangle DBE$ and place them back-to-back as shown below. Mark the midpoints of sides $\overline{AC}$, $\overline{CE}$, $\overline{ED}$, and $\overline{DA}$, and then connect them. Prove that the connected midpoints form a square.



Some suggestions:

- Add segments $\overline{AE}$ and $\overline{CD}$ to the picture. Show that two sides of the "square" are parallel to $\overline{AE}$ and the other two are parallel to $\overline{CD}$.
- If you rotate $\triangle CBD$ by 90° clockwise about point B, where does point C land and where does point D land? What does this tell you about the lengths of $\overline{AE}$ and $\overline{CD}$ and the angle between them?

Investigation 4.13

HISTORICAL PERSPECTIVE: PARALLEL LINES

In Investigation 4.12, you proved the Parallel and Side-Splitting Theorems. These proofs used certain facts about parallel lines that you will learn more about here, all in the historical context of the mathematician Euclid.

AXIOMS AND EUCLID'S *THE ELEMENTS*

A *deductive proof* is an argument that shows how one statement follows logically from other accepted statements. And how did these "other statements" become accepted? Well, they too follow logically from still other accepted statements. This can't go on forever, though. At some point, people have to agree that they will accept some statements without proof. These statements are called *axioms*. These are statements so obvious that they require no proof. Put another way, these are statements that are assumed "for the sake of argument"—"if you'll agree to something for a moment, I'll show you how something else follows directly from it."

You may be familiar with the term *postulate* from other *Connected Geometry* modules or other mathematics courses. *Postulate* and *axiom* are two names for the same thing.

Due to extensive work begun in the late 19th century, every branch of mathematics can now be presented by writing down a set of axioms and making deductions from them. The most influential forerunner of this work is *The Elements* of Euclid (c. 300 B.C.), a collection of 13 books, each devoted to some aspect of Greek mathematics. Little is known about Euclid as a person. He lived and worked in Alexandria, a Greek city on the northern coast of Africa in what is now Egypt, at the mouth of the Nile River.

An English translation of Euclid's work is *The Elements* (Dover Publications, 1956).

Everything in Euclid's *The Elements* was supposed to be derived from a small number of definitions and axioms. *The Elements* contains five main geometric axioms:

Axiom 1 Exactly one straight line can be drawn between any two points.

Axiom 2 A line segment can be extended indefinitely in both directions.

Axiom 3 A circle can be drawn with any center and any radius.

Axiom 4 All right angles are congruent.

Axiom 5 If two lines in a plane are cut by a transversal, and if the sum of the angles on one side of the transversal is less than 180°, then the two lines will eventually intersect on the side where the sum of the angle measures is less than 180°.

HISTORICAL PERSPECTIVE: PARALLEL LINES

(continued)

The last of Euclid's axioms listed on the previous page (known as "Euclid's fifth") was very controversial. Some people neither thought it was obvious nor were willing to accept it for the sake of argument. Many people thought it could be proved using the other four axioms. Indeed, it can't—a fact established in the mid-19th century.

An example of a commonly-held notion in Euclid's work is "The whole is greater than any of its parts."

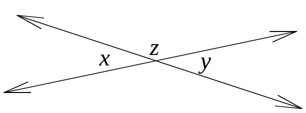
Starting from these five axioms (and other "commonly-held notions"), Euclid's *The Elements* derives all of the classical results of geometry and arithmetic. *The Elements* contains a proof of the Pythagorean Theorem, derives the formulas for the areas of polygons and circles, shows that every whole number can be factored into primes, and shows how to find the greatest common divisor for any two whole numbers.

Because *The Elements* was the first such undertaking in human history, it did contain errors. Indeed, we now know that Euclid needed many more definitions and axioms than were written down in *The Elements* in order to make his deductions possible. The glorious fact remains, however, that *in essence*, Euclid got it right in every respect!

The next few problems invite you to look at some of the results that can be obtained using the axioms in Euclid's *The Elements*.

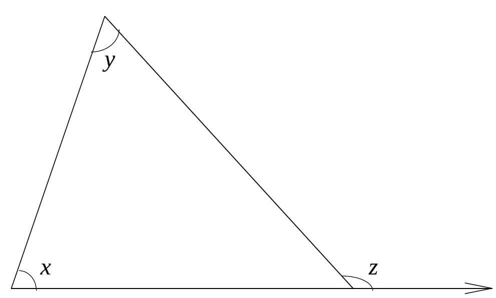
1. For each of the five axioms in Euclid's list, draw a diagram that illustrates it, and explain the axiom in your own words.

2. If two lines intersect, two pairs of vertical angles are formed. Prove that vertical angles are congruent.

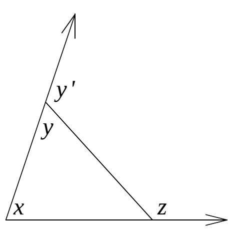


What's the relation of x to z? Of y to z?

3. An "exterior angle" of the triangle is formed by one side of a triangle and the extension of another:



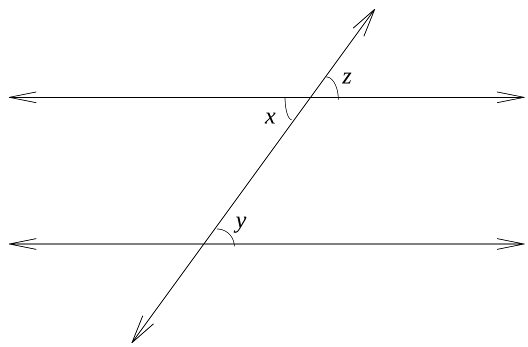
z is the measure of an exterior angle.



Hint: If $z < y$, then $z + y' < 180^\circ$ (why?).

Using only Euclid's five axioms (and other simple "commonly-held notions"), show that an exterior angle of a triangle has a measure that is greater than or equal to the measures of either nonadjacent interior angle of the triangle. (In the figure, show that $z \geq y$ and $z \geq x$.)

4. Suppose that two parallel lines are cut by a transversal:



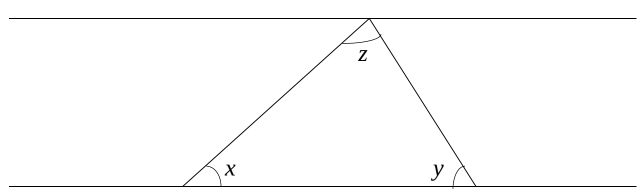
Hint: If *x* and *y* are not equal, one is larger than the other.

a. Angles like those with measures x and y are called *alternate interior angles*. Show that alternate interior angles are congruent.

b. Explain why angles with measures z and y (which are called "corresponding angles") are congruent.

5. Use what you have shown so far to prove that the sum of the angle measures in a triangle is 180°.

Hint:



The two horizontal lines are drawn parallel.

6. Use the result of Problem 5 to show that the measure of an exterior angle of a triangle is, in fact, *equal to the sum* of the measures of the nonadjacent interior angles. Explain why the results from Problem 3 can now be strengthened to: $z > y$ and $z > x$.

7. Prove the converse of the theorem in Problem 4:

> If two lines form congruent alternate interior angles with a transversal, then the two lines are parallel.

Hint: If the lines aren't parallel, they intersect. Together with the third line, you would see a triangle. Now use Problem 6.

PERSPECTIVE ON ERATOSTHENES

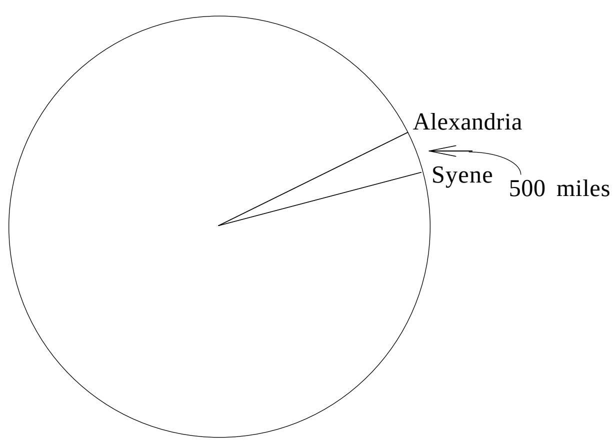
This essay explains how an ancient Greek mathematician was able to estimate the circumference of the Earth. How accurate was his estimate?

The Greek mathematician Eratosthenes devised an ingenious way to answer a perplexing question: How can one measure the circumference of our vast Earth?

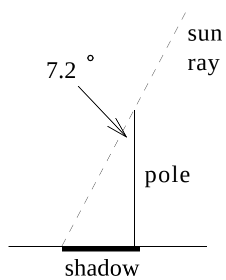
Eratosthenes was an Alexandrian who lived around 200 B.C. (a century after Euclid). He was a contemporary of Archimedes and the chief librarian at the great library in Alexandria. Eratosthenes' method for estimating the Earth's circumference uses some of the facts you've proved about parallel lines. Here's what he did:

These calculations are given in terms of miles and degrees. Eratosthenes used other units of measurement.

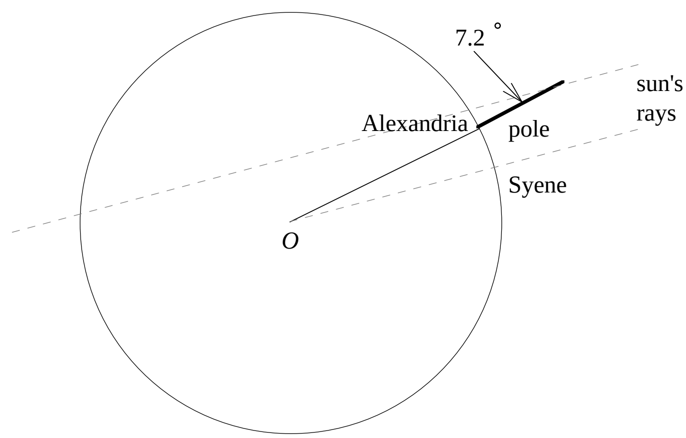
Alexandria was about 500 miles due north of another town, Syene (now called Aswan). Eratosthenes imagined slicing the Earth through the two cities, like this:



On the first day of summer each year, Eratosthenes knew that the sunlight reflecting into a well in Syene shot right back up into an observer's eyes at noon. He concluded that, on that day, at that time, the center of the Earth, Syene, and the sun were in a straight line. Eratosthenes also assumed that the sun was so far away that its rays were essentially parallel by the time they hit the face of the Earth. Finally, on that day, a vertical pole erected at Alexandria cast a shadow making an angle of 7.2°, as shown in this picture:



The pole at Alexandria



8. Using what you know about parallel lines, what is the measure of the central angle at O?

9. Using the result of Problem 8 and the fact that a circle has 360°, how much of the Earth's circumference is taken up between Syene and Alexandria? Use this to estimate the circumference of the Earth.

10. Check your estimate against a current estimate of the Earth's circumference (from an atlas or encyclopedia). How close were you and Eratosthenes?

You can read more about the connections between mathematics, the Earth, and astronomy in the book *Poetry of the Universe* by Robert Osserman (Anchor Books, 1995).

Investigation 4.14

Defining Similarity

In this section of the module, you will learn about an important geometric idea called *similarity*. After defining similarity in terms of scaling, you will develop tests for similar triangles. For example, if the three angles of one triangle are congruent to the three angles of another triangle, the triangles are not necessarily congruent, but are they always similar? Finally, you will use relationships in similar triangles to solve many kinds of problems in which you need to find an unknown distance or length.

Words like *enlargements*, *reductions*, *scale factors*, and *dilations* are some of the terms you've met again and again in this module. The common theme uniting them is called *similarity*. By taking a picture and enlarging or reducing it, you create another picture that is *similar* to the first, so here is one way to define "similar" in geometry:

Definition

Two figures are ***similar*** if one is a scaled copy of the other.

You can also use the word "dilation" to define "similar." Here's a try:

Definition

Two figures are ***similar*** if one is a dilation of the other.

To test this definition, look at a picture of the head of Trig the horse. Trig is accompanied here by his little sister, Girt. She's smaller than Trig, but shares all of his features:

A family portrait

1. Is Girt's head a dilated copy of Trig's? If so, find the center of dilation.

2. Is the picture of Girt similar to the picture of Trig?

Here's another family portrait of Trig and Girt, this time in a different pose:

3. Is the picture of Girt still similar to the picture of Trig?

4. Can you still dilate one picture onto the other? Explain.

5. Expand the dilation definition of "similarity" so that even these pictures of Trig and Girt can be called similar.

For Discussion

Here are some suggestions for ways to define similarity using dilation terminology. Do these definitions solve the problem you had with the second family portrait of Trig and Girt? Are these definitions equivalent?

DEFINITION

Two figures are ***similar*** if you can rotate and/or flip one of them so that it can be dilated onto the other.

DEFINITION

Two figures are ***similar*** if one is congruent to a dilation of the other.

CHECKPOINT

6. If two figures are congruent, are they similar? Explain. If two figures are similar, are they congruent?

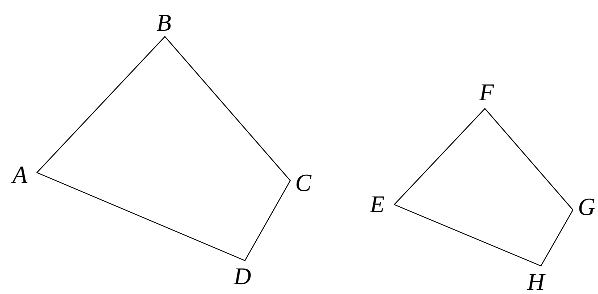
7. If two figures are similar to the same figure, are they similar to each other? Explain.

FOR DISCUSSION

Compare the mathematical meaning of the word "similar" to its use in everyday life. Are the meanings related?

SOME NOTATION

The symbol for similarity is the upper part of the symbol for congruence: $\sim$. Thus the statement "$ABCD \sim EFGH$" is read "*ABCD* is similar to *EFGH*" and means that the two polygons below are scaled copies of each other.



$ABCD \sim EFGH$

The similarity symbol, $\sim$, means "has the same shape as," and the congruence symbol, $\cong$, means "has the same shape *and* has the same size as."

KEEPING TRACK OF ORDER

For congruent triangles, the statement $\triangle ABC \cong \triangle XYZ$ conveys specific information about their *corresponding* parts. Namely,

$$\angle A \cong \angle X, \quad \angle B \cong \angle Y, \quad \text{and} \quad \angle C \cong \angle Z$$

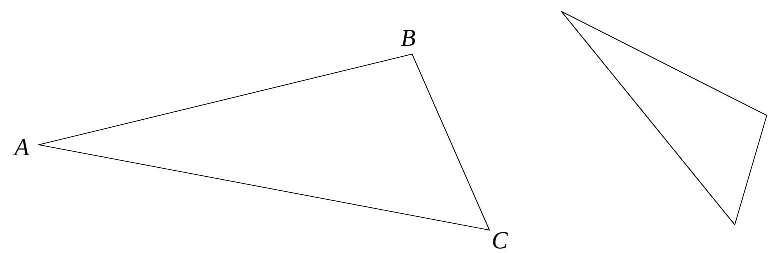
and

$$\overline{AB} \cong \overline{XY}, \quad \overline{BC} \cong \overline{YZ}, \quad \text{and} \quad \overline{AC} \cong \overline{XZ}.$$

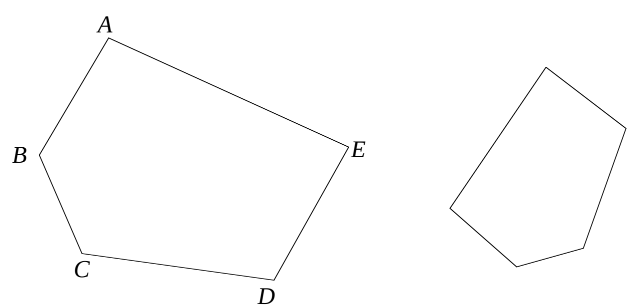
For figures that are similar, the order of the letters representing their vertices also gives specific information about corresponding parts.

8. In the pictures below, $\triangle ABC \sim \triangle XYZ$ and $ABCDE \sim PQRST$. Copy the pictures and label the vertices of $\triangle XYZ$ and $PQRST$ correctly.

a.



b.



SIMILAR TRIANGLES

In Problem 9 of Investigation 4.5, you wrote several ways to test if two triangles were similar (only then, you used the phrase “scaled copies”). One of these ways probably was:

> *Two triangles are similar if their corresponding angles are congruent and their corresponding sides are proportional.*

For future reference, let’s refer to this as the “Congruent Angles, Proportional Sides” test.

Using the word “dilation,” we have another test for similar triangles:

Two triangles are similar if one is congruent to a dilation of the other.

Let’s refer to this as the “Congruent to a Dilation” test.

9. Suppose that $\triangle NEW \sim \triangle OLD$. If $m\angle N = 19^\circ$ and $m\angle L = 67^\circ$, find the measures of all the other angles.

10. If $\triangle ABC \sim \triangle DEF$, which of the following must be true?

a. $\frac{AB}{DE} = \frac{BC}{EF}$

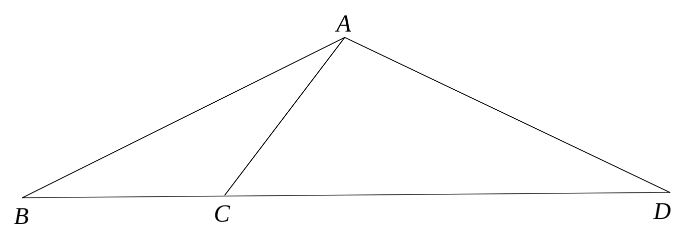
b. $\frac{AC}{BC} = \frac{DF}{EF}$

c. $\frac{BC}{AB} = \frac{DF}{DE}$

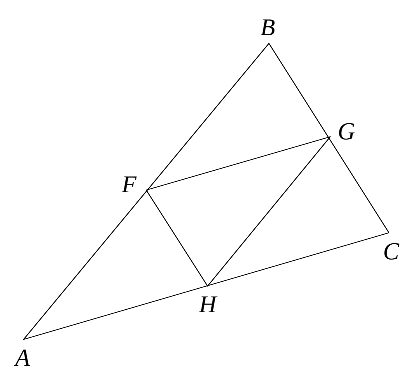
d. $AC \times DE = AB \times DF$

11. In the figure below, $\triangle ACB \sim \triangle BAD$. Explain why $\triangle ACB$ is isosceles.

Hint: Look for congruent angles.



12. In the picture below, F, G, and H are midpoints of the sides of $\triangle ABC$. Show that $\triangle ABC \sim \triangle GHF$.

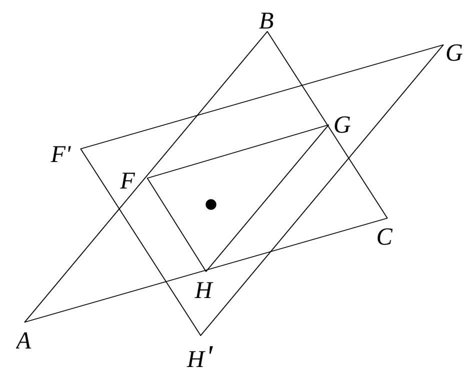


WAYS TO THINK ABOUT IT

Since F, G, and H are midpoints of the sides of $\triangle ABC$, the Side-Splitting Theorem tells us that the sides of $\triangle GHF$ are parallel to the sides of $\triangle ABC$. And then, by the Parallel Theorem, each side of $\triangle GHF$ is half as long as the corresponding side of $\triangle ABC$ (why?).

Why did we pick 2 as the dilation factor?

So, let's dilate $\triangle GHF$ by a factor of 2. Any center of dilation will do; let's use some point inside the triangle:



All the sides of the dilated triangle $\triangle G'H'F'$ are twice as long as the corresponding sides of $\triangle GHF$. Thus, they are equal in length to the sides of $\triangle ABC$ (why?). That is,

$$AB = G'H'$$
$$AC = G'F'$$

and

$$BC = H'F'.$$

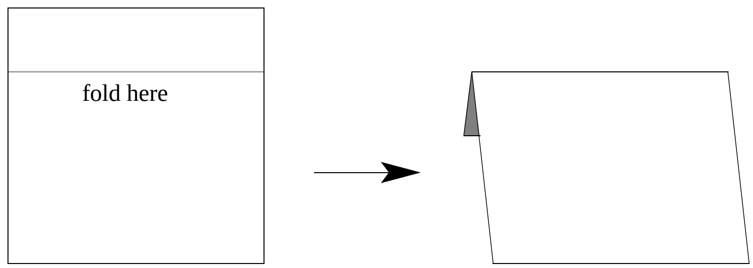
So, $\triangle ABC \cong \triangle G'H'F'$ by SSS. Then, $\triangle ABC$ is congruent to a dilation of $\triangle GHF$, making $\triangle ABC \sim \triangle GHF$.

In the next investigation, you will develop some similarity theorems about triangles that will reduce this whole proof to just a few lines.

TAKE IT FURTHER

This is a great question to spring on a friend while you're eating at a restaurant or in the school cafeteria. Use a napkin (which is often square) to pose the challenge.

13. Take a square sheet of paper and fold one fourth of it behind. You will be left with a rectangle. The challenge is to fold the paper to form a rectangle that's similar to this one but has half its area. No fair using a ruler!



Hint: Even though you've folded part of the square behind, you can still unfold it and work with the entire square.

Investigation 4.15

SIMILAR TRIANGLES

When you studied congruent triangles, you spent a good deal of time establishing minimum conditions necessary to guarantee that two triangles were congruent. For example, the "side-angle-side" postulate, abbreviated by the letters SAS, says:

> Two triangles are congruent if two sides and the included angle of one triangle are congruent to two sides and the included angle of the other.

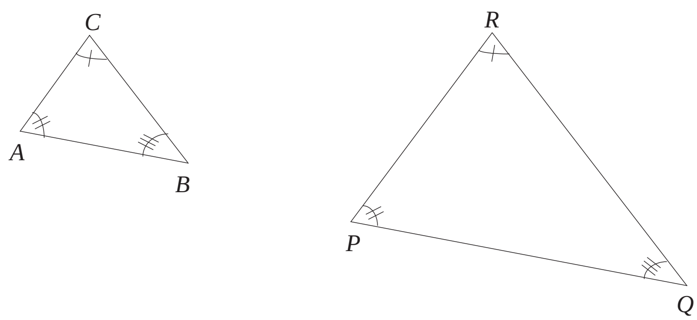
Tests like this allow you to check whether two triangles are congruent without having to move them around to see if all their parts coincide. It would be nice to have comparable tests for similar triangles. Let's see what you can find.

1. The main tests for triangle congruence are SAS, ASA, AAS, and SSS. Are there equivalent tests for triangle similarity? Here are some possibilities to consider. For each test, draw a pair of triangles that share the attributes listed. Then check to see if they're similar. Also, see if you can find a counterexample.

- **a.** If two triangles have all three corresponding angles congruent, the triangles are similar (AAA similarity).
- **b.** If two triangles have a pair of corresponding side lengths proportional and a pair of corresponding angles are congruent, the triangles are similar (SA similarity).
- **c.** If two triangles have two pairs of sidelengths proportional and the included angles are congruent, the triangles are similar (SAS similarity).
- **d.** If two triangles have all three corresponding sidelengths proportional, the triangles are similar (SSS similarity).

A FIRST TRIANGLE SIMILARITY THEOREM: AAA

Suppose two triangles, $\triangle ABC$ and $\triangle PQR$, have congruent corresponding angles. Can you prove that they are similar?



One definition of similar triangles is:

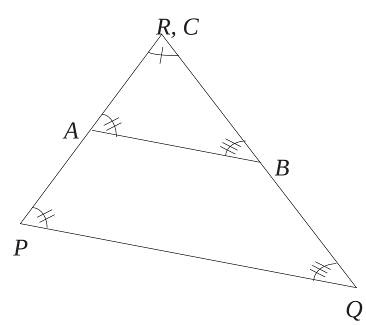
> *Two triangles are similar if their corresponding angles are congruent and their corresponding sides are proportional.*

Let's use this definition to write a proof.

CONGRUENT ANGLES, PROPORTIONAL SIDES:

In earlier investigations, you checked to see if triangles were scaled copies (similar) by placing one inside the other to form a pair of nested triangles. Let's try placing $\triangle ABC$ inside $\triangle PQR$ so that $\angle C$ and $\angle R$ coincide:

Why do the triangles line up this way?



What fact are we using about parallel lines here?

Since $\angle A$ is congruent to $\angle P$, we know that $\overline{AB} \parallel \overline{PQ}$. Then, by the Parallel Theorem, $\frac{RP}{CA} = \frac{RQ}{CB} = \frac{PQ}{AB}$. Altogether then, we know that the angles of $\triangle ABC$ are congruent to the angles of $\triangle PQR$, *and* their corresponding sides are proportional. Thus, the triangles are similar.

THEOREM 4.3 *AAA Similarity*

If two triangles have corresponding angles congruent, the triangles are similar.

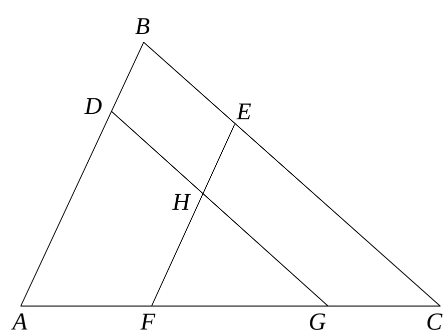
2. In a sense, the requirement that all three angles of one triangle be congruent to the corresponding three angles of the other is overkill. Why?

3. Rewrite the theorem to give the *minimal* angle conditions necessary for two triangles to be similar.

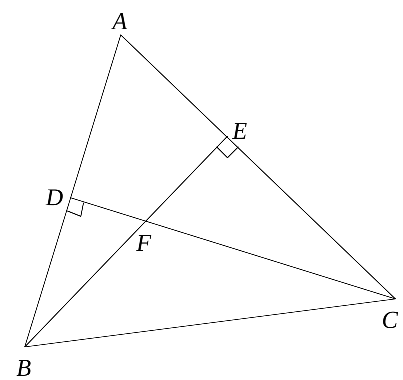
4. Does it make sense to have an ASA test for triangle similarity? Explain.

5. Are two quadrilaterals similar if the angles of one are congruent to the corresponding angles of another? Either prove this as a theorem or disprove it by finding a counterexample.

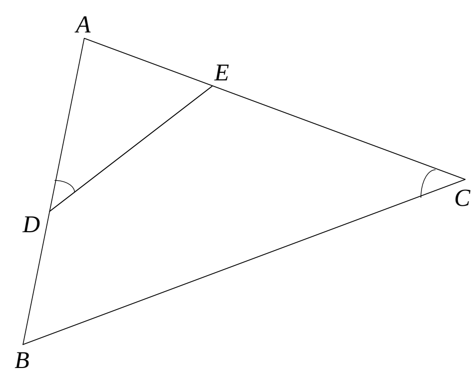
6. In the figure below, $\overline{DG} \parallel \overline{BC}$ and $\overline{EF} \parallel \overline{BA}$. Prove that $\triangle ABC \sim \triangle FHG$.



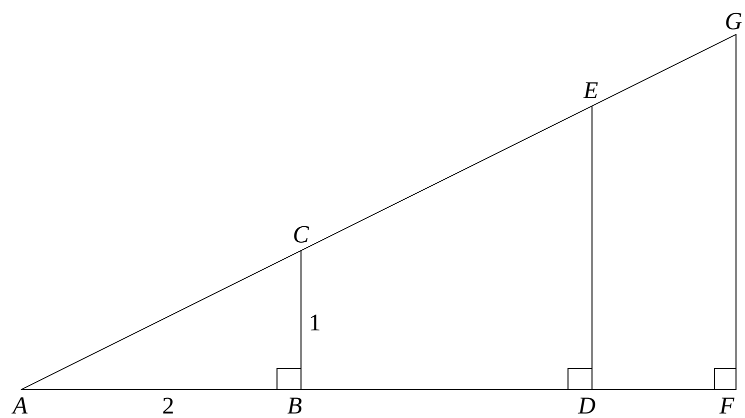
7. $\overline{BE}$ and $\overline{CD}$ are altitudes of $\triangle ABC$ below. List as many pairs of similar triangles as you can find and explain why the triangles are similar.



8. In the figure below, $\angle ADE \cong \angle ACB$. Explain why $\triangle ADE \sim \triangle ACB$.



9. In the figure below, $AB = 2$ and $BC = 1$. Without making any measurements, find the values of $\frac{AD}{DE}$ and $\frac{AF}{FG}$. Explain how you got your answers.



10. Draw a right triangle ABC and the altitude from the right angle to the hypotenuse. The altitude will divide $\triangle ABC$ into two smaller right triangles.

   a. There are two pairs of congruent angles (other than the right angles) in your picture. Find and label them.

   b. Make a copy of your triangle. Then cut out the two smaller right triangles. Position them in such a way as to convince yourself that they are similar to each other and to $\triangle ABC$.

   c. Explain why all three of these triangles are similar.

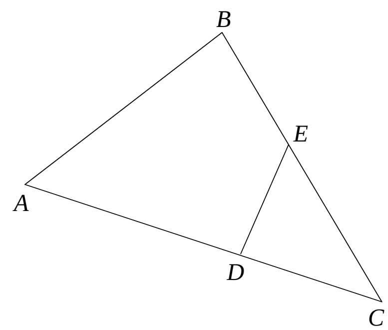
A SECOND TRIANGLE SIMILARITY THEOREM: SAS

THEOREM 4.4 *SAS Similarity*

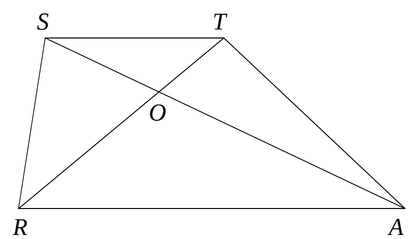
If two triangles have two pairs of sidelengths proportional and the included angles are congruent, the triangles are similar.

11. You can prove this theorem by following the same basic setup used for AAA similarity. Write a proof of SAS similarity, and present it to your class.

12. In the figure below, $AB = 4$, $BC = 5$, $AC = 6$, $DC = 2.5$, and $EC = 3$. Prove that $\triangle ABC \sim \triangle EDC$ and find the length of $\overline{DE}$.



13. *RATS* is a trapezoid with $\overline{RA} \parallel \overline{ST}$ and diagonals $\overline{RT}$ and $\overline{AS}$ meeting at O:



a. Explain why $\triangle ROA \sim \triangle TOS$.

b. From part a, you can say that $\frac{RO}{TO} = \frac{OA}{OS}$. Why?

c. Given the proportion from b and the fact that $\angle ROS \cong \angle TOA$, Eloida claims that $\triangle ROS \sim \triangle TOA$ by the SAS similarity test. Is this true? Explain.

A THIRD TRIANGLE SIMILARITY THEOREM: SSS

THEOREM 4.5 *SSS Similarity*

If two triangles have all three pairs of corresponding sidelengths proportional, the triangles are similar.

14. If you try to prove this theorem using the same method as the AAA and SAS proofs, something doesn't quite work. Look back at the AAA and SAS proofs. In both of them, you placed $\triangle ABC$ inside $\triangle PQR$ so that $\angle C$ and $\angle R$ coincided. Can you do the same here?

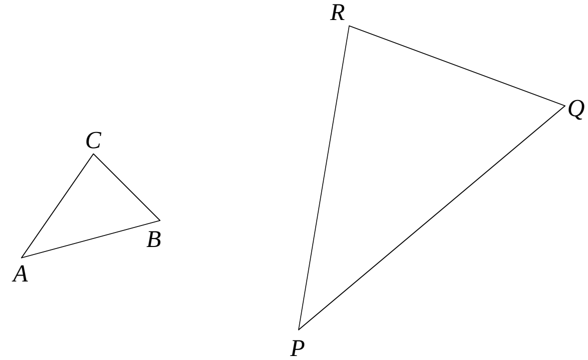
Since the "congruent angles, proportional sides" method isn't well suited to proving SSS, let's use a different definition of triangle similarity:

Two triangles are similar if one is congruent to a dilation of the other.

Here's a proof of SSS using this definition.

CONGRUENT TO A DILATION

Suppose the corresponding sides of $\triangle ABC$ and $\triangle PQR$ are proportional.



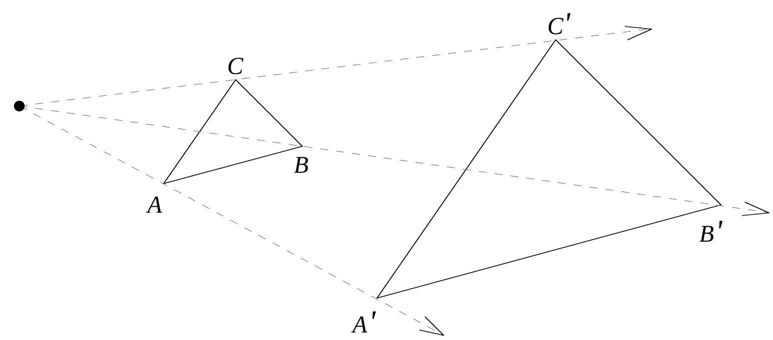
This means there is some positive number k for which

$$PQ = k \times AB$$
$$QR = k \times BC$$

and

$$RP = k \times CA.$$

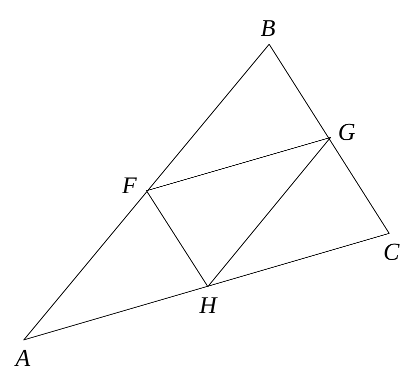
Let's dilate $\triangle ABC$ by k, picking any point as the center of dilation:



The dilated triangle $\triangle A'B'C'$ has sides that are congruent to the corresponding sides of $\triangle PQR$ (why?). Then, by the SSS triangle congruence test, $\triangle A'B'C' \cong \triangle PQR$. Thus, $\triangle PQR$ is congruent to a dilation of $\triangle ABC$, and the two triangles are similar.

You solved this problem in Investigation 4.14 (Problem 12) in a different way. What have you learned since then that makes this problem easier to solve?

15. In the picture below, F, G, and H are midpoints of the sides of $\triangle ABC$.



Show that $\triangle ABC \sim \triangle GHF$.

16. In Problem 15, there are several other pairs of similar triangles. List them and explain why they are similar.

17. The sides of a triangle are 4, 5, and 8. Another triangle is similar to it, and one of its sides has length 3. What are the lengths of its other two sides? Is there more than one possible answer?

18. A triangle has sides of lengths 2, 3, and 4 inches. Another triangle is similar to it, and its perimeter is 6 inches. What are the sidelengths of this triangle?

CHECKPOINT

19. $\triangle JKL$ has $JK = 8$, $KL = 12$, and $JL = 16$. Points M and N are on $\overline{JK}$ and $\overline{KL}$ respectively, with $JM = 6$ and $LN = 9$.

a. Explain why $\overline{MN} \parallel \overline{JL}$.

b. Prove that $\triangle MKN \sim \triangle JKL$ using each of the following theorems:

i. AA

ii. SAS

iii. SSS

20. $\triangle DEF$ is similar to $\triangle ABC$ and has sides that are three times as long. Find the numerical values of the following ratios, and name the triangle similarity theorem that allows you to draw your conclusion.

a. The ratio of any two corresponding altitudes

b. The ratio of any two corresponding angle bisectors

c. The ratio of any two corresponding medians

TAKE IT FURTHER

21. If the altitudes of one triangle are congruent to the corresponding altitudes of another triangle, prove that the triangles are congruent.

Some suggestions: Make a sketch of the two triangles and their altitudes. You should be able to find three ways to express the area of each triangle. Use this along with the SSS similarity test to first prove that the triangles are similar. Then show (using the results from Problem 20a) that the triangles are, in fact, congruent.

Investigation 4.16

USING SIMILARITY

Calculating Distances and Heights

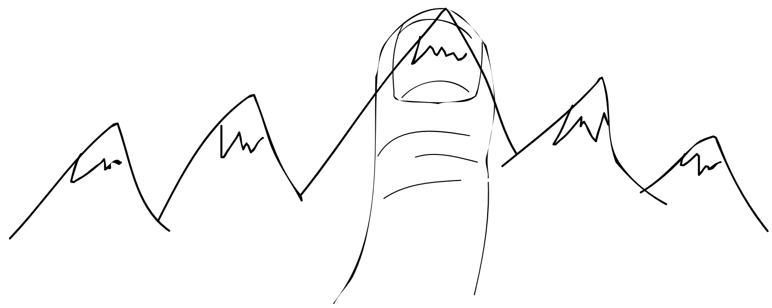
Over the ages, people have developed clever methods to answer the questions, “How far away is that?” and “How tall is that?” This investigation focuses on ways that similarity can answer these distance and height questions.

A SEA STORY

Several years ago, two friends were sailing off Old Orchard Beach in Maine. The sky was so clear that they could see the faraway top of Mount Washington in northern New Hampshire, towering some 6600 feet above sea level. One of the two sailors held her left arm straight out in front of her in a “thumbs up” gesture to get an idea of their distance from the base of the mountain.

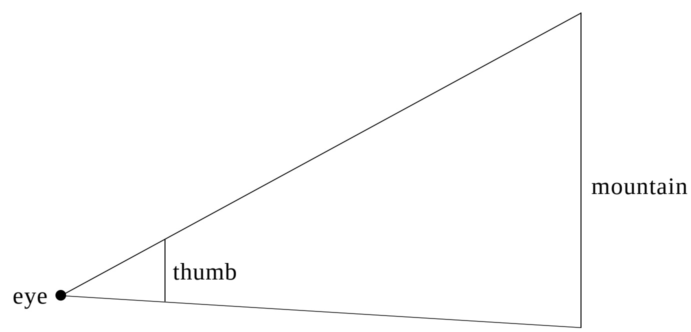
She positioned herself so that she could see how much of her thumb would cover the mountain. When the top of her thumbnail lined up exactly with the top of the

mountain, one of the wrinkles on her thumb lined up with the shore at sea level—the whole 6600 feet on a thumb!



As she held her outstretched arm very still, her companion measured the distance from her eye to the place on her thumb that lined up with the edge of the shore. Then they measured the length of the part of her thumb that had covered Mount Washington. Using similar triangles, they calculated their distance to the base of the mountain. "You know," they remarked later, "that calculation turned out to be surprisingly accurate!"

1. The picture below shows a rough sketch of the situation. What assumption is the picture making? Where is a pair of similar triangles?



Remember that the height of the mountain is 6600 feet.

2. If the length of your thumb covering the mountain is 1 inch, and the distance from your eye to the bottom of your thumb is 14 inches, calculate your distance from the mountain.

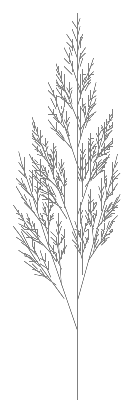
3. Measure the length of your thumb or your index finger and the distance from the base of the finger to your eye when your arm is fully extended. If you know the height of any object that can be covered by that finger, you can then determine your distance from that object. Pick an object and use this technique to figure out how far away it is. Check your results by measuring the actual distance.

If necessary, you can estimate what your finger covers. If a building is 60 meters high and your finger covers about one third of it, then the covered region is about 20 meters.

A "SHADY" METHOD

Using shadows is a quick and reasonably accurate way to find the heights of trees, flagpoles, buildings, and other tall objects. To begin, measure the length of the shadow your object casts in sunlight. Also measure the shadow cast at the same time of day by a yardstick (or some other object of *known* height) standing straight up on the ground.

Now that you know the lengths of the two shadows and the length of the yardstick, you can use the fact that the sun's rays are approximately parallel to set up a proportion with similar triangles.



4. Suppose that a tree's shadow is 20 feet long and the yardstick's shadow is 17 inches.

a. Draw a picture that shows the tree, the yardstick, the sun's rays, and the shadows.

b. Find a pair of similar triangles and explain why they are similar.

c. How tall is the tree?

5. Use the shadow method to find the height of some tall object for which you can obtain the actual height. Record the details of your measurements and prepare a presentation for your class. By how much did the result of your calculations differ from the actual height? What might cause these differences?

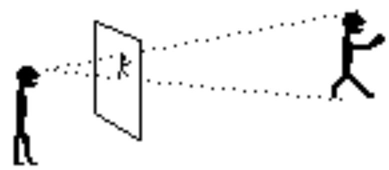
TINY PLANETS

In 1996, the very first discovery of planets that were not part of our own solar system was reported. Astronomers had good reason to believe such planets existed, but for the first time it became possible to find them. The newspapers gave accounts of the clever techniques that astronomers used to find these planets, despite the fact that the planets themselves had not been seen. No report seemed to explain *why* the planets were such a challenge to see, but that is something you can figure out using similarity.

As of early 1996, there were no reports of planets of the nearest star. Suppose, though, that it *did* have a planet the size of the Earth. How large would that planet *appear* to the unaided eye?

First, you must figure out what that question means! Imagine looking at a person through a window and seeing how much of the window that person fills. Better yet, try tracing the outline of the person on the window.

6. If you actually make the tracing, you can measure the height of the image, but you can also figure it out if you have enough information about the height of the person, the distance of the person from you, and the distance of the window from you. Here is a picture of that situation. Carefully describe how you could use similar triangles to figure out how tall the image on the window would be.



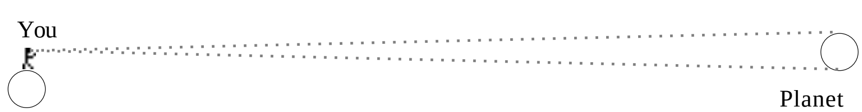
7. Apply your theory. Imagine seeing a five-foot tall person standing outside your window about 30 feet away from you. Imagine you are about two feet from the window. About how tall will the image be?

8. Now imagine viewing the distant planet through the same window. Before you can figure out how tall the image will be, what information must you have?

9. The nearest star to us (other than our own sun!) is about four light-years away. A "light-year" is a unit of distance—specifically the distance that light travels in one year. Light travels at 186,000 miles per second. How far does it travel in one year? How far away is the nearest star?

The diameter of the Earth is approximately 7900 miles.

10. If a planet that is the same distance away as the star in Problem 9 has the same diameter as the Earth, how large will its image appear on a window two feet from you?



Use $\frac{1}{20}$ of an inch for the width of an "o."

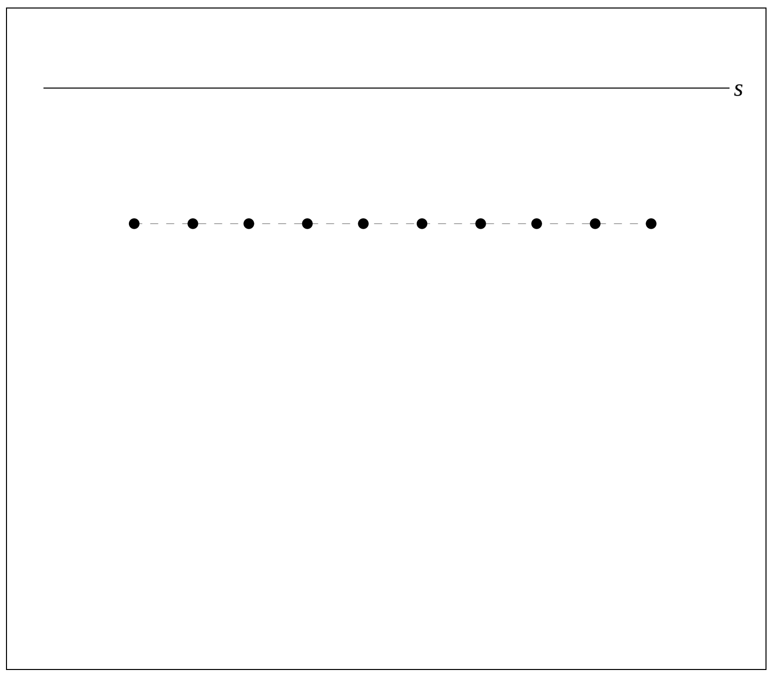
11. About how much must that image be magnified to be as big as an "o" on this page?

Segment Splitters

Here are two intriguing experiments to try, both of which depend on similarity.

EXPERIMENT ONE: THE PROJECTION METHOD

Ask your teacher for the blackline master page that accompanies this investigation. The page, as shown below, contains a segment *s* and ten equally-spaced points below it, all lying on a segment parallel to *s*.

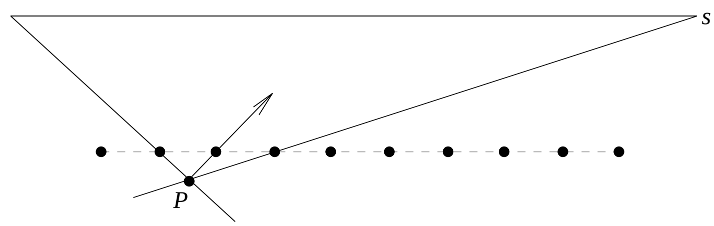


You can enlarge this picture if you don't have the blackline master from the *Teaching Notes*.

Think of a "straightedge" as a ruler without any marks on it. The edge of a stiff sheet of paper or cardboard works well.

Choose any three consecutive points on the dotted line. Using a straightedge, draw a line connecting the left endpoint of segment *s* with the leftmost of the three points.

Also, draw a line connecting the right endpoint of segment s with the rightmost point. In the example below, the lines meet at point P:



Now draw a line through point P and the middle point. Extend your line far enough so that it intersects segment s.

Notice that your line passes right through the middle of segment s. You've found the midpoint of segment s *without taking any measurements*.

12. Start with a new copy of the blackline master. Now split segment s into three congruent pieces without taking any measurements. Compare your work with that of your classmates to see if they did it the same way.

13. Divide segment s into five congruent pieces. Divide another segment into seven congruent pieces.

14. Why do you think this experiment is called "The Projection Method"?

EXPERIMENT TWO: THE PARALLELS METHOD

For this method, you will need a straightedge, a sheet of lined notebook paper, a blank overhead transparency provided by your teacher, and an erasable marker.

15. Use the marker and the straightedge to draw a segment on the transparency. Now, with nothing other than your lined notebook paper and marker, divide the segment in half.

16. Draw another segment on the transparency, and divide it into three congruent pieces.

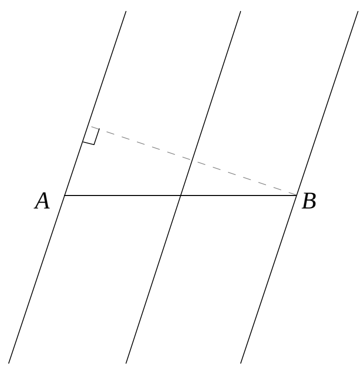
17. Draw some more segments on the transparency, and divide them into five and seven congruent pieces. What's the largest number of divisions you can make with your notebook paper?

18. Explain your method to a classmate.

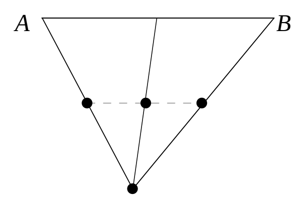
WHY DO THE METHODS WORK?

The pictures below show examples of using the parallels method and the projection method. In both cases, the segment $\overline{AB}$ is being split in half:

The parallels method has a dashed segment added to the picture to help you.



Parallels Method



Projection Method

19. Use properties of similar triangles to show that both methods do indeed split segment $\overline{AB}$ in half.

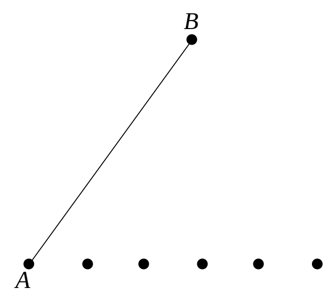
TAKE IT FURTHER. . . . .

20. Extend your argument to show that the segment-splitting methods work for *any* division of a segment into congruent parts.

USING SOFTWARE GEOMETRY

With geometry software, you can model the parallels method and add a nice feature that's not possible on paper: after you've divided a given segment into a number of congruent parts, it will stay equally divided even as you shorten or lengthen it.

To divide a segment $\overline{AB}$ into five congruent parts, begin by translating point A by any fixed distance five times so that A and the five new points are collinear:



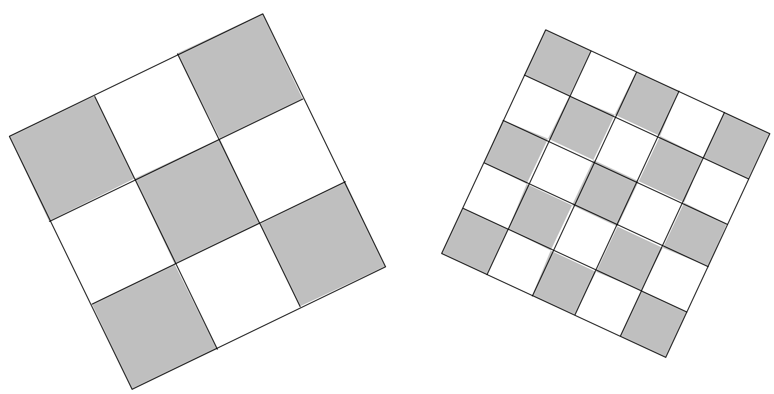
$\overline{AB}$ should remain equally divided when you move either of its endpoints around the screen.

21. Using the parallels method, complete the construction so that $\overline{AB}$ is divided into five congruent parts. Start by figuring out how and where to make the parallel lines. How should they be oriented—to what will they be parallel?

22. Experiment with this method by drawing some more segments with the software and dividing them into as many congruent parts as you like.

TAKE IT FURTHER. . . . .

23. Use software to create checkerboard patterns like the ones below. Your patterns should be scalable—dragging on a vertex of the outer square should cause the whole figure to shrink or to grow.



PERSPECTIVE ON SARAH MARKS

This essay discusses the life and work of Sarah Marks and focuses on the segment–splitting device that she patented.

Segment-splitting devices date back at least to the time of Euclid. In fact, London's Science Museum contains replicas of bronze splitting devices found at Pompeii. Closer to the present, the scientist Sarah Marks (1854–1923) invented and patented her own device for splitting a segment into any number of equal parts.

Sarah Marks' father died when she was seven. For many years afterward, she helped support her mother, sister, and six brothers. At the age of nine, Marks went to London to be privately educated at a school run by two of her aunts. There she was tutored in Latin, Greek, French, Hebrew, the classics, music, art, and mathematics, not only by her aunts and uncles, but by several of her cousins as well.

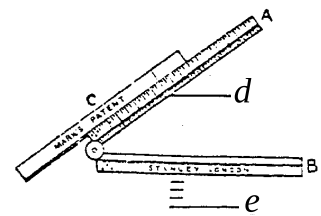
Marks' inventions include a device for measuring a person's pulse. Her work with electrical arcs led to improved searchlights and movie projectors. She also devoted much of her time to the women's suffrage movement and to lowering the barriers that women faced in education, laboratories, and scientific societies. She was the first woman to be nominated to become a Fellow of the Royal Society of London and was awarded the society's Hughes Medal for originality in research.

Marks' segment-splitter patent is shown on the next page. See if the description given in the patent provides enough information for you to figure out how the device works.

MARKS'

PATENT LINE DIVIDER.

For Dividing any space in a number of equal parts.



DESCRIPTION.—A B a hinged rule with firm joint, the limb A fitted to slide in an undercut groove upon the plain rule C. C has needle points on the under side to prevent it from slipping when placed in any position. The limb A of the rule is divided on both edges into eights, quarters, half-inches and inches, which are consecutively numbered so that any set may be taken.

TO USE THE LINE DIVIDER.—Suppose the space *d* to *e* is to be divided into any number of parts—say thirteen : Taking the half-inch line, hold the rule B on the line *e* and open the rule A until the division marked 13 on the inside edge is coincident with the line *d*; now notice that the single line on rule C is opposite the 13, and in this position press it down so that the needle points on the under side get sufficient bite to prevent it slipping; placing the fingers firmly on C, slide the part A upwards so that it may stop consecutively opposite each of the 13 divisions, as indicated opposite the line on the rule C, a pencil line drawn along B across *d e* at each stoppage opposite the numbers 12, 11, 10, 9, &c., will give the required divisions. To produce the lines in ink, the rule, after setting, may be moved to the upper line first, and the division lines be drawn downwards.

The rule may be worked in any direction for drawing line, vertical, horizontal, or oblique, and for any division of a space from 2 to 80 parts. It will be found convenient to Architects and Engineers for dividing any spaces without previous trial—such as treads and risers of stairs, joists, roof-timbers, girders, brick spaces, for drawing section line shading, &c., and will be found a saving of time for division of a space into 3 parts and upwards.

For open spaces, multiples of the space must be taken. In very close divisions, the joint may interfere with the last 2 or 3 divisions, in this case a line must be drawn at each setting and produced after the rule is removed.

SOLE AGENT—

W. F. STANLEY,

5, Great Turnstile, Holborn, London, W.C.

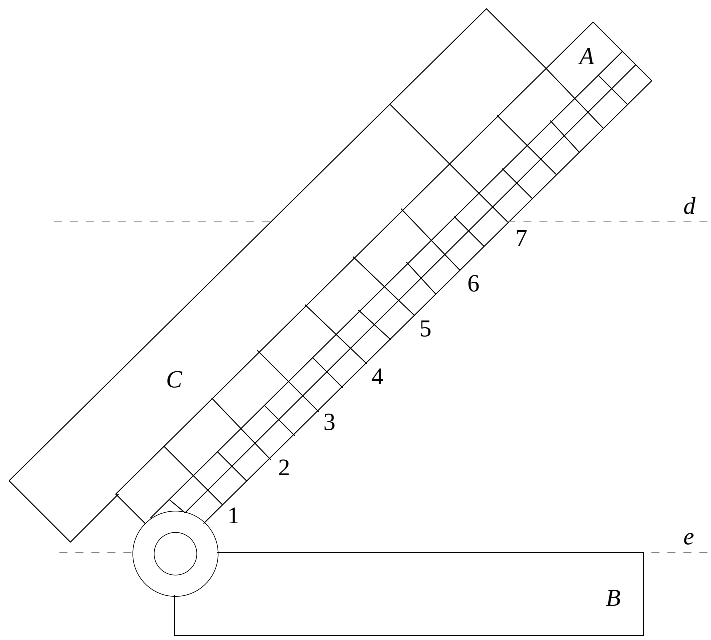
PRICE 5/- IN CASE.

SOME MORE EXPLANATION

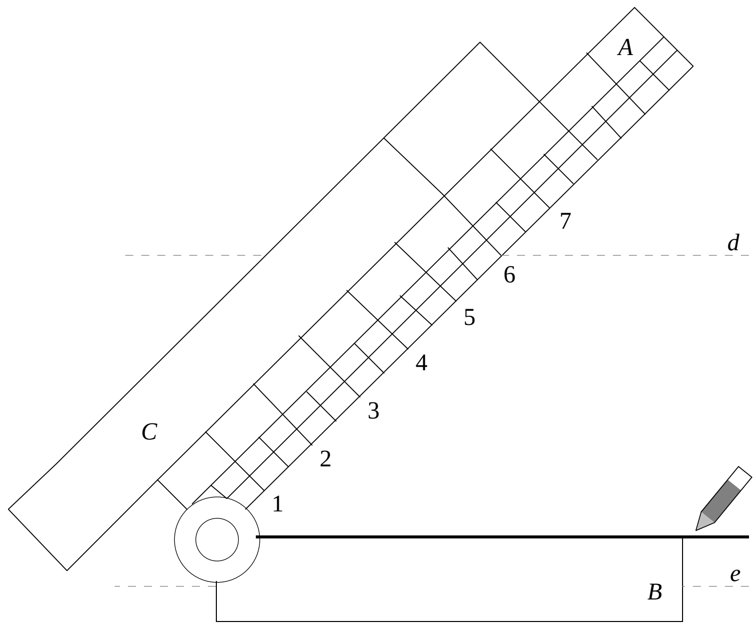
Marks' segment splitter consists of three arms, *A*, *B*, and *C*. Arms *A* and *B* are hinged together, and arm *C* slides along a grooved track on arm *A*. Arm *C* contains just one mark, whereas arm *A* is marked off into 80 equal parts. (Fewer are shown in the figure below.) The underside of arm *C* has two protruding pins that can be pressed into a piece of paper to prevent the device from slipping.

To divide the region between lines *d* and *e* into seven equal parts, place arm *B* along line *e* and open arm *A* until the seventh mark touches line *d*. Then slide arm *C* until its mark coincides with this seventh mark. The pins on arm *C* are now pressed down to keep it in place.

A wing nut keeps the angle formed by arms *A* and *B* fixed.

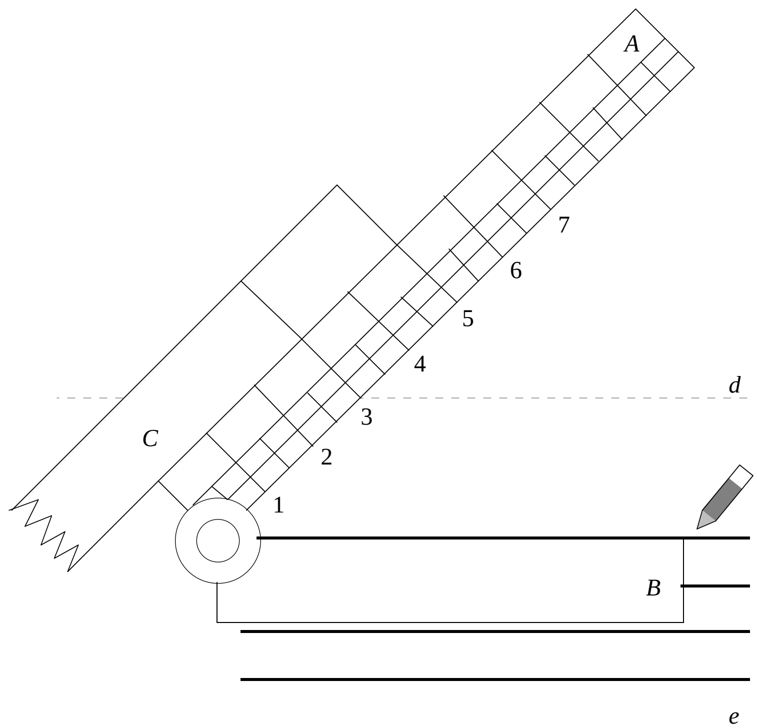


Without moving arm *C*, slide arm *A* until its sixth mark aligns with the single mark on arm *C*. As arm *A* slides up, arm *B* moves up also, maintaining the same angle between *B* and *A*. Draw a line along the top of arm *B*.



Continue sliding arm *A* up, one mark at a time. For each new position, draw a line along the top of arm *B*. The picture below shows the device after the first four divisions of the region between lines *d* and *e* have been made.

Arm *C* extends into the margin.



24. Explain why Marks' segment splitter works.

25. In the example above, Marks' splitter divides the region between lines *d* and *e* into seven equal parts. Explain how you would use the device to divide a *segment* into seven equal parts.

26. Does Marks' method for splitting a segment seem similar to any of the other segment-splitting methods you worked with earlier? Explain.

27. **Project** Build a model of Marks' segment splitter.

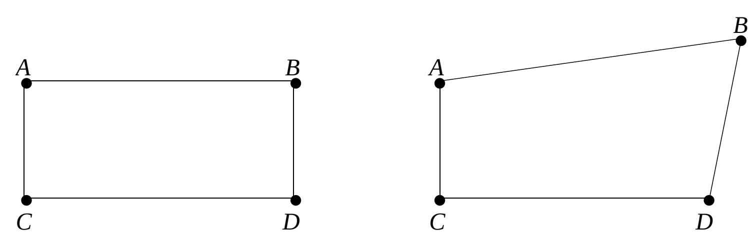
A Constant-Area Rectangle

Similarity can help you find the heights of tall objects and calculate distances from faraway places. But there's more to similarity than just measurement applications. This investigation and the next one, "The Geometric Mean," explore a connection between similarity and the area of rectangles.

CONSTRUCTING A RECTANGLE

Have you ever made a rectangle with geometry software? One way is to use the segment tool to draw four segments, and then fiddle with the angles to make sure they measure 90°.

This method of *drawing* a rectangle works, but if you tug on a side or a vertex like B, the figure becomes a mere quadrilateral:



A rectangle? *Nope!*

By comparison, a rectangle that is *constructed* behaves much better. No matter which side or vertex you tug, the rectangle might change its dimensions, but it remains a rectangle.

28. Use geometry software to construct a rectangle.

29. Sometimes, people refer to constructed rectangles as "UnMessUpable." Why is this an appropriate name for them?

30. Use the software to calculate the area of your rectangle. Drag a vertex of the rectangle so that its dimensions change. Does its area change as well?

WHAT IS A CONSTANT-AREA RECTANGLE?

While exploring rectangles with geometry software, we became intrigued by the following possibility: Is there a way to construct a rectangle so that its perimeter can change when you drag a vertex, but its *area must remain the same*?

If there is a way to build such a rectangle, we could call it a "constant-area rectangle."

The length and width do not have to be integers.

31. Give length and width dimensions of four rectangles that all have an area of 36 square feet.

32. If l represents the lengths of the rectangles in Problem 31 and w represents their widths, what is the relationship between l and w?

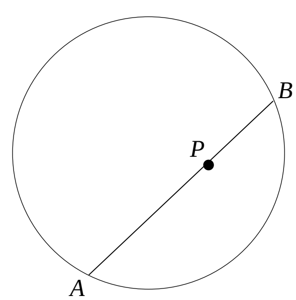
33. How many rectangles are there with an area of 36 square feet?

One of the special pleasures of mathematics comes from finding an unexpected connection between two topics that seem to have nothing in common. The section below contains problems that may, at first, seem like a detour from the constant-area rectangle. But as you work through it, ask yourself if any of the results can help you to construct a constant-area rectangle.

THE POWER OF A POINT

Try this with geometry software.

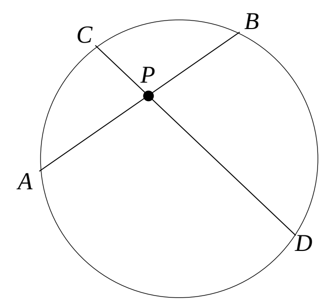
Draw a circle and pick a point P anywhere inside it. Then draw a chord of the circle that passes through P. Point P divides the chord into two segments: $\overline{PA}$ and $\overline{PB}$. Measure the lengths of these segments and calculate the product $PA \times PB$.



34. Draw other chords through point P and calculate the same product. Record any of your observations.

These products have a special name. They're called the *power of point P*.

35. The circle below has two chords, $\overline{AB}$ and $\overline{CD}$, that intersect at point P.



a. What can you predict about the lengths PA, PB, PC, and PD?

Hint: $\angle ACD$ **and** $\angle ABD$ **intercept the same arc.**

b. Add segments $\overline{AC}$ and $\overline{BD}$ to the illustration. The first step in proving your conjecture from part a is to show that $\triangle APC \sim \triangle DPB$. Explain why these two triangles are indeed similar.

c. Use the fact that $\triangle APC \sim \triangle DPB$ to write a proportion that includes PA, PB, PC, and PD. Rearrange the proportion to prove your conjecture from part a.

d. How does this result prove that the product of the chord lengths is the same for *any* chord through P?

Making the Connection

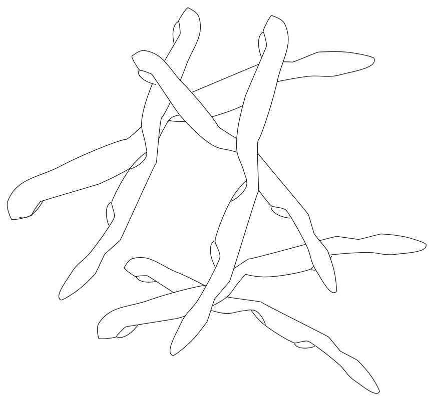
Using your power-of-a-point findings, you can construct constant-area rectangles. The first scenario below connects the power-of-a-point results to constant-area rectangles made from wooden sticks. The second scenario extends the idea to geometry software.

Scenario One Suppose that you decide to build a collection of equal-area rectangles out of long wooden sticks. Each rectangle is to have an area of 12 square feet, and no two rectangles can have the same dimensions. You can, of course, measure the

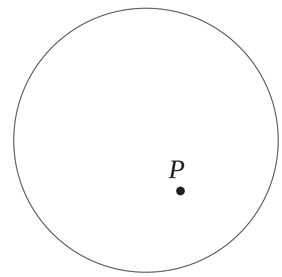
sticks and then cut them at appropriate places. Measuring becomes tedious though, especially if you plan on building lots of rectangles.

36. How can the setup below help you to construct your rectangles?

The actual circle and sticks would be larger.



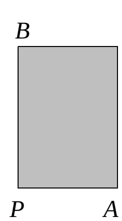
a few of your wooden sticks

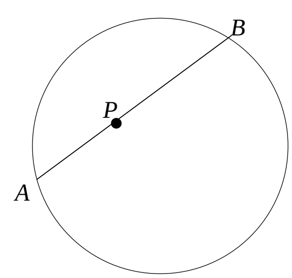


power of point P = 12

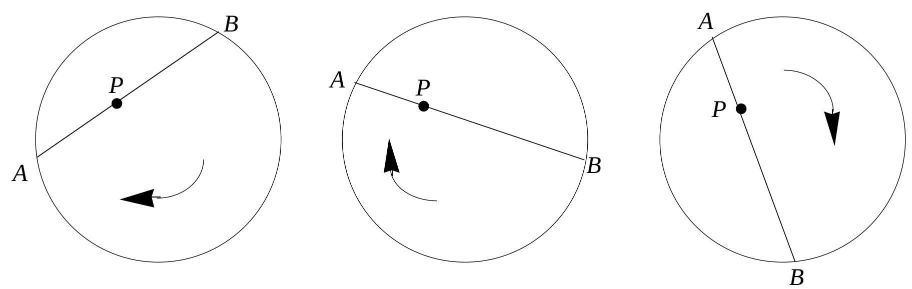
Scenario Two The wooden stick construction gave you a convenient way to build some rectangles that shared the same area. "Not bad," you say, "but I want more of them, and cutting wood is too tiring!" So you decide to create an animated movie, called *A Rectangle of Constant Area*. You'll be making this "movie" with geometry software, and you would like whoever watches the movie to see the following:

To begin, the computer screen shows a rectangle on one side and the power-of-a-point construction on the other. The length *PA* and the width *PB* of the rectangle are linked to the corresponding chord segments and are equal to them. So when the chord segment lengths change, the rectangle's dimensions will, too.





As you move point A around the circle, the chord $\overline{AB}$ spins, always passing through the stationary point P:



37. Use geometry software to build a construction like the one described here. Your construction might include an animation feature that allows point A to travel automatically around the circle.

38. As chord $\overline{AB}$ spins, describe what happens to the rectangle.

39. The purpose of this geometry software construction was to create a movie called *A Rectangle of Constant Area*. Explain why you think this construction does or does not satisfy this goal.

Be sure to provide popcorn.

40. Show your "movie" to someone else and explain how it works.

41. Does your construction show *all* possible rectangles that share the same area? Explain.

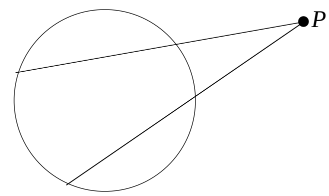
TAKE IT FURTHER

42. Draw a circle, a point P within it, and calculate the power of P.

a. Are there other locations for P within the same circle that have the same power? Find a few and explain your reasoning.

b. Find the locations of *all* points within your circle that have the same power as P. What does this collection of points look like?

43. Explore the power-of-a-point construction for points that lie *outside* the circle. Can you still find a constant product?



44. For a constant-*perimeter* rectangle, the area of the rectangle can change, but the perimeter must remain the same when you drag a side or vertex. Use geometry software to construct a constant-perimeter rectangle.

The Geometric Mean

In the previous section of this investigation, “A Constant Area Rectangle”, you used the power-of-a-point construction to build a collection of rectangles that all shared the same area. By exploring this construction some more, you will learn about the concept of *geometric mean*.

COMPLETING A RECTANGLE

The figure below shows a complete rectangle on the left and only the length of another rectangle on the right.

No measuring, please!

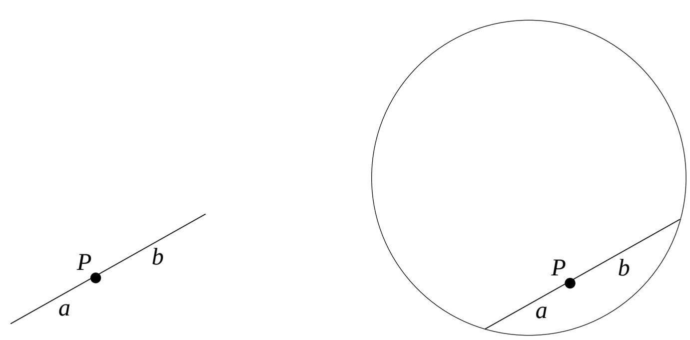
45. Use the power-of-a-point construction to construct the missing width of the rectangle so that both rectangles have the same area.

WAYS TO THINK ABOUT IT

Here’s one way to do the construction:

How can you draw a circle through two points?

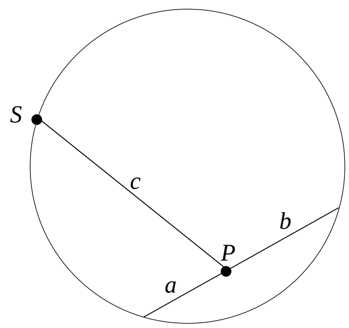
- Draw segments a and b so that they form a longer segment and meet at a point P. Then draw a circle with this new segment as a chord:

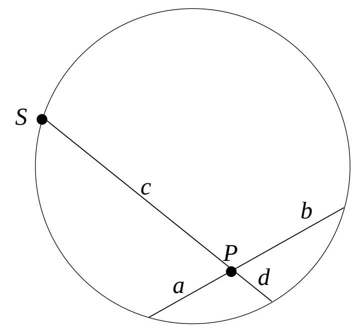


How can you locate point *S*?

Is it possible that no point *S* on the circle gives a length $PS = c$? If so, when would this happen? What could you do to solve the problem?

- Find a point S on the circle such that $PS = c$. Then, extend $\overline{PS}$ so that it intersects the circle. The length d is the missing width of the rectangle.

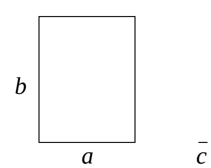




FOR DISCUSSION

Explain why this construction method works.

46. Once again, begin with the $a \times b$ rectangle from Problem 45. Explain how to draw a rectangle with the same area, only this time, its length must be the very tiny segment c:

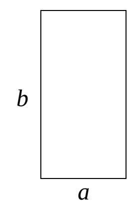


Can you use the same circle as above to complete this construction? What difficulties do you encounter? Explain how to redraw the circle so that the construction works.

The Rectangle and the Square

Still another rectangle challenge for you to try:

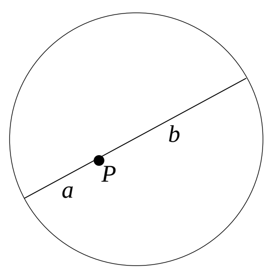
47. Use the power-of-a-point construction to draw a square with the same area as this rectangle:



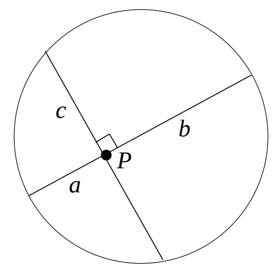
Ways to Think About It

Here's one way to do the construction:

- Draw segments of lengths a and b so that they meet at a point P to form a longer segment. Then draw a circle with this new segment as a diameter:



- Draw a chord through P perpendicular to the diameter. The segment with length c shown below is a side of the desired square.



48. Write and Reflect Explain why this construction method works.

49. Write and Reflect What is the algebraic relationship between a, b, and c?

The length c has a special name:

DEFINITION

If $c^2 = ab$ (or equivalently, $c = \sqrt{ab}$), then c is called the ***geometric mean*** of a and b.

50. Compute the geometric mean of each pair of numbers.

a. 2 and 8

b. 3 and 12

c. 4 and 6

d. 5 and 5

WAYS TO THINK ABOUT IT

The *arithmetic mean* (or *average*) of two numbers a and b is defined as $\frac{a+b}{2}$. So the arithmetic mean of 6 and 24 is $\frac{6+24}{2} = 15$. The mean 15 is midway between 6 and 24:

$$15 - 6 = 9$$
$$24 - 15 = 9.$$

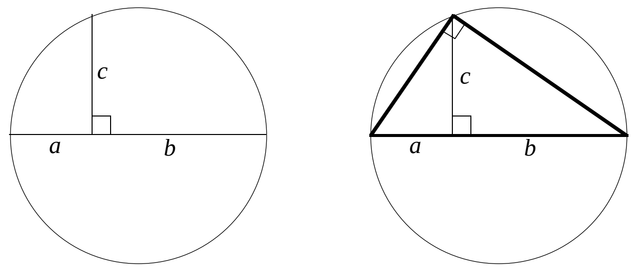
The *geometric mean* of two numbers defines a different kind of "midway" point between them. To get from 6 to 24, you can multiply by 2 ($6 \times 2 = 12$) and then by 2 again ($12 \times 2 = 24$). Thus 12, the midway point in your journey, is the geometric mean of 6 and 24. Another way to say that 12 is midway between 6 and 24 is to write a proportion:

$$\frac{6}{12} = \frac{12}{24}.$$

$$6 \xrightarrow{+9} \underset{\text{Arithmetic Mean}}{15} \xrightarrow{+9} 24$$

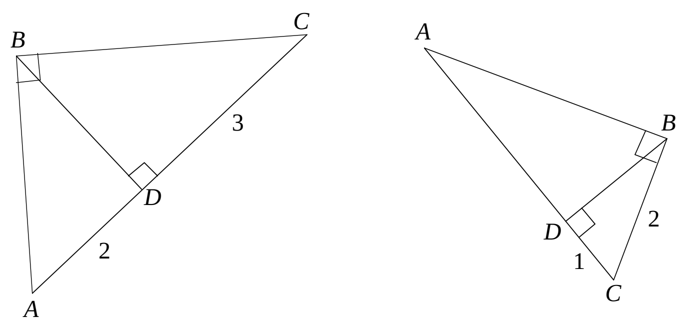
$$6 \xrightarrow{\times 2} \underset{\text{Geometric Mean}}{12} \xrightarrow{\times 2} 24$$

51. If you add two extra segments to the geometric mean picture on the left, then three right triangles are formed in the semicircle. Why?



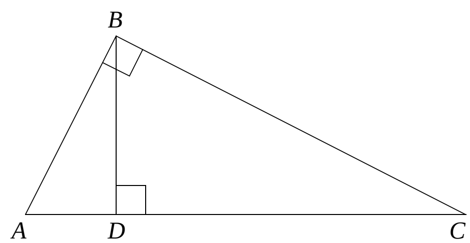
Usually, just the right triangles are shown and not the circle that is used to construct them. In the next problem, the circles were erased after constructing the right triangles.

52. Find all the unknown lengths of the segments in these two figures:

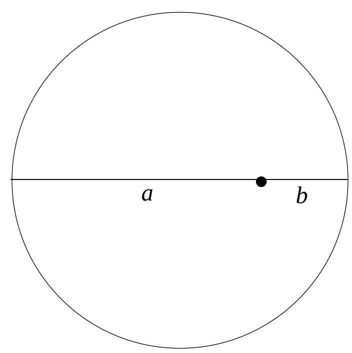


53. Construct a segment whose length is the geometric mean of a 1-inch segment and a 3-inch segment.

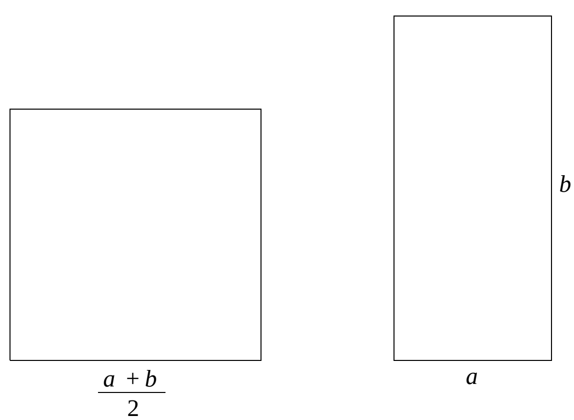
54. Construct a proof that BD is the geometric mean of AD and DC by showing that $\triangle ADB \sim \triangle BDC$ and writing an appropriate proportion.



55. **a.** Copy the figure below, and draw a segment whose length is the geometric mean of the segments of lengths a and b shown below. Also draw a segment whose length is their arithmetic mean. Which is longer? Does this relationship always hold? Is the arithmetic mean ever equal to the geometric mean?



b. Here is a picture of a square and a rectangle:



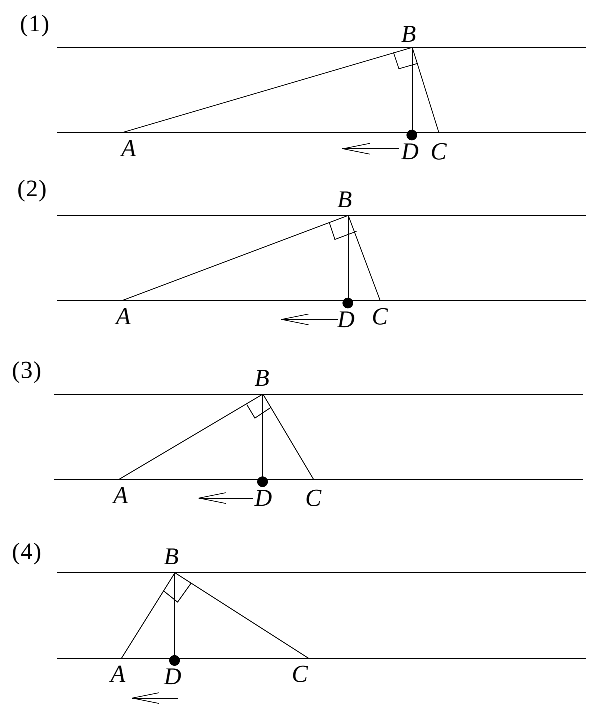
i. Which has the larger perimeter?

ii. Which has the larger area?

ANOTHER RECTANGLE-OF-CONSTANT-AREA "MOVIE"

In the previous investigation, you used geometry software to build a "movie" about a constant-area rectangle. This movie showed a continuous sequence of rectangles, all with the same area. To create this effect, you used the power-of-a-point construction to control the movement and dimensions of the rectangles.

There is also a way to make the movie using the geometric mean construction. The illustration below shows four frames from a geometric mean construction as point D moves to the left and point A remains stationary.



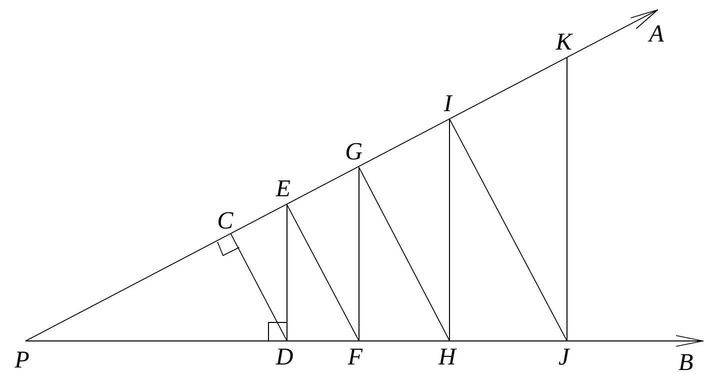
56. There are two lengths whose product remains the same throughout each of these four frames. Which are they?

57. Use this setup to build another *Rectangle-of-Constant-Area* movie.

58. Your constant-area-rectangle movies show rectangles that range from narrow and tall to wide and short. Which of the two constructions—the power of a point or the geometric mean—seems to generate a larger range of constant-area rectangles? Why?

TAKE IT FURTHER. . . . .

59. $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are rays that meet at point P. The segments $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, . . . form a zigzag pattern and alternate between being perpendicular to $\overrightarrow{PA}$ and $\overrightarrow{PB}$.



If $PC = 1$ and $PD = x$, find the length of the following segments (it helps if you do them in order):

a. $\overline{PE}$;

b. $\overline{PF}$;

c. $\overline{PG}$;

d. $\overline{PH}$.

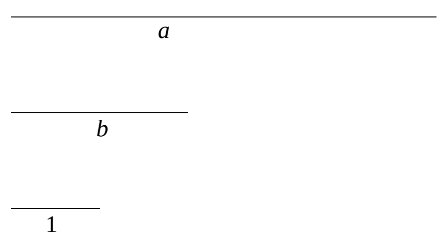
Hint: Recall the meaning of *geometric sequence*. **(See p. 25 in Investigation 4.3.)**

Do you detect a pattern here? Without doing any more calculations, give the lengths of $\overline{PI}$, $\overline{PJ}$, and $\overline{PK}$.

No Measuring, Please!

Below are three segments of lengths a, b, and 1. We can call a segment of length 1 a *unit segment*. For each of the problems, can you construct a segment with the given length?

All of the constructions can be done with just a straightedge, a compass, and some lined notebook paper. Resist the urge to measure with a ruler—you won't need to! To construct some of these lengths, you'll need to apply concepts that you've learned about similar triangles, segment splitters, the power of a point, and the geometric mean.



60. $a + b$

61. $a - b$

62. $3a$

63. $2a - b$

64. ab

65. $\frac{a}{b}$

66. $\frac{b}{3}$

67. a^2

68. $\sqrt{b}$

69. $\sqrt{ab}$

70. $\sqrt{a^2 + b^2}$

71. $\sqrt{a^2 - b^2}$

72. $\frac{a+\sqrt{b}}{2} + \frac{a-\sqrt{b}}{2}$ (Yikes! Or maybe not . . .)

Investigation 4.17

AREAS OF SIMILAR POLYGONS

In this section of the module, you will explore what happens to the areas of polygons, "blobs," and circles when you scale them. Along the way, you will derive the area and circumference formulas for circles and discover some fascinating facts about π.

How do the areas of similar polygons compare? Simple polygons like rectangles are a good place to start the investigation.

1. Draw a rectangle and then scale it by a factor of 2.

a. How do the dimensions of the original rectangle compare with the dimensions of the scaled one?

b. How many copies of the original rectangle fit into the scaled one?

c. How does the area of the scaled rectangle compare to the area of the original one?

2. Draw a rectangle and then scale it by a factor of $\frac{1}{3}$.

a. How do the dimensions of the original rectangle compare with the dimensions of the scaled one?

b. How many copies of the scaled rectangle fit into the original one?

c. How does the area of the scaled rectangle compare to the area of the original one?

3. State and prove a theorem that starts like this:

> If rectangle $ABCD$ is scaled by a factor of r to get rectangle $A'B'C'D'$, then the area of $A'B'C'D'$ . . .

4. If two triangles are similar, and the scale factor is r, we know that the ratio of the lengths of two corresponding sides is r. Show that:

a. The ratio of their perimeters is r.

b. The ratio of the lengths of two corresponding altitudes is also r.

c. The ratio of their areas is r^2.

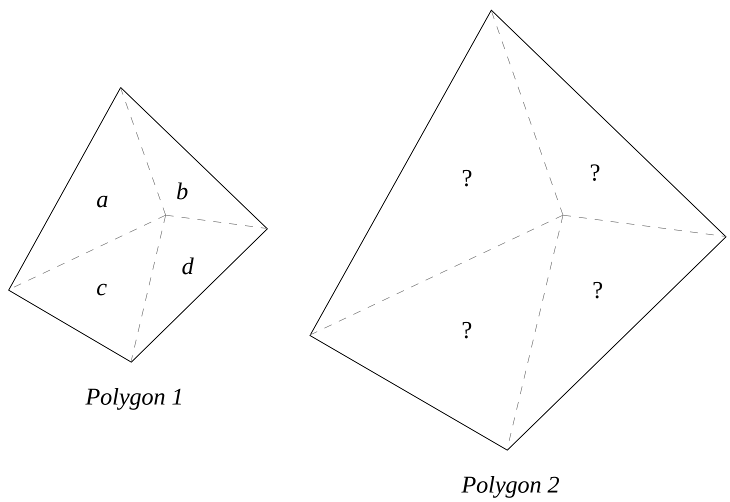
5. One side of a triangle has length 10, and the altitude to that side has length 12. If all the sides of the triangle are tripled, what is the area of the new triangle?

6. According to Problem 4, if you scale any triangle by a factor of 4, then 16 copies of it should fit inside the scaled copy. Check this out with paper and scissors or geometry software.

7. According to Problem 4, if you scale a triangle by $2\frac{1}{2}$, then $6\frac{1}{4}$ copies of the original should fit inside the scaled copy (why?). Check this out with paper and scissors or geometry software.

Now that you've calculated the areas of similar rectangles and triangles, take a look at similar polygons with *any* number of sides:

8. In the figure below, Polygon 1 was scaled by a factor of r to obtain Polygon 2. Polygon 1 is divided into four triangles with areas of a, b, c, and d.



a. What are the areas of the corresponding triangles in Polygon 2?

b. What is the total area of Polygon 2?

c. What is the total area of Polygon 1?

Use these results to complete the following theorem:

THEOREM 4.6

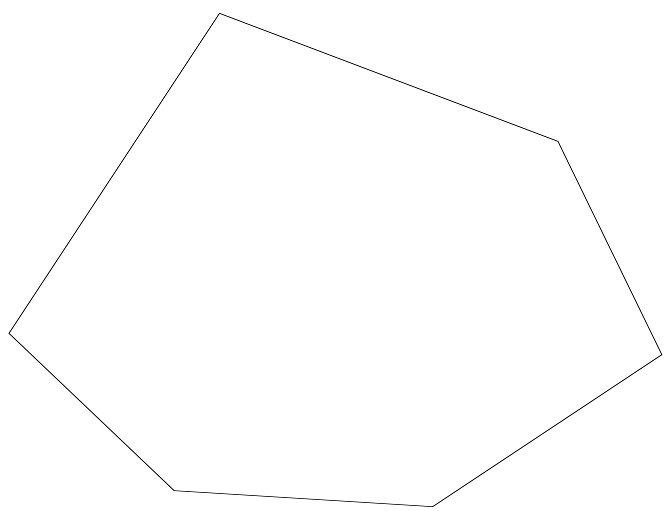
If a polygon is scaled by some positive number r, then the ratio of the area of the scaled copy to the original is . . . .

9. Carefully prove this theorem, justifying each step.

10. Hans has two cornfields that he wants to plant. One measures $400' \times 600'$ and the other measures $200' \times 300'$. Bessie Moonfeed, the owner of the grain store, says, "The big field will take eight bags of seed. The small field has sides half as big, so you'll need four more bags for that. Will that be cash or charge?" A few days later, Hans returns to the grain store very upset. Why?

Try dividing it into triangles.

11. **a.** Carefully trace this polygon and estimate its area:



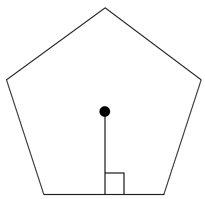
b. If you scale the polygon by a factor of 1.5, what will the new area be?

12. One square has an area that is 12 times the area of another. What is the ratio of

a. their sides?

b. their diagonals?

INTRODUCING THE APOTHEM



An apothem of a regular pentagon

The *apothem* of a regular polygon is a perpendicular segment from the center point of the polygon to one of its sides.

13. **a.** Why doesn't it make a difference which side you choose when drawing an apothem?

b. Show that the area of a regular polygon is equal to half the product of its perimeter and apothem length.

The result of the last problem is important enough to record as a theorem:

THEOREM 4.7

The area, A, of a regular polygon is equal to half the product of its perimeter, P, and the length of its apothem, a. In symbols:

$$A = \frac{1}{2}Pa.$$

14. Use the area formula $A = \frac{1}{2}Pa$ to calculate the area of a square whose sidelength is 12. Check your result by calculating the area of the square another way.

15. What is the area of a regular hexagon whose sidelength is 8?

CHECKPOINT

16. A rectangle is scaled by a factor of $\frac{1}{4}$. Compare the area of the scaled rectangle to the area of the original one.

17. A triangle is scaled by a factor of 5. Compare the area of the scaled triangle to the area of the original one.

18. A polygon has an area of 17 square inches. If it's scaled by a factor of 2, what is the area of the new polygon?

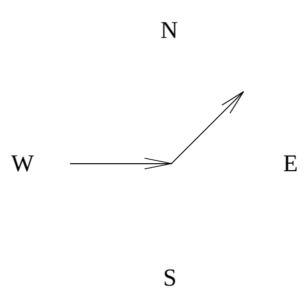
TAKE IT FURTHER. . . . .

As part of its new budget, your town decides to build a recreation center to benefit the community. You're elected as the student representative from your school to assist with the planning and development.

The town committee has narrowed down the choices for the recreation center to two possible locations. Both sites are located in the same neighborhood, and each spot seems appealing. Also, the price of the two land parcels is roughly the same. However, you're not sure whether both spaces cover the same amount of land. You'd like to pick the parcel with the larger piece of land and calculate its area so that the architect can begin to design the specifications of the recreation center.

One possible method for finding the areas is to walk along the border of each property and record the length of each side as well as how much you turn at each corner. In fact, these are often the kinds of measurements recorded on land deeds.

Here's the description of your walk for both parcels of land:



An example of a northeast turn

- **Parcel 1** Start at one corner of the land. Walk east 798 feet, turn northeast and walk 543 feet. Head north for 678 feet, and then walk along a straight line back to where you started.
- **Parcel 2** Start at one corner of the land. Walk 884 feet west, turn north and walk 442 feet. Then walk due east for 554 feet, turn northwest and walk 61 feet. Head east for 718 feet, and then walk along a straight line back to where you started.

Now it's up to you to calculate the area of each land parcel as precisely as possible. You'll probably want to make a drawing of each space using either a ruler and protractor or geometry software. Here are some questions to ask yourself that might help:

- How can I draw each parcel so that its distance and angle measurements are preserved?
- How can I fit the drawing onto a single sheet of paper or computer screen?
- Can I calculate the area of the whole space at once, or should I divide it into smaller pieces?

19. Prepare a presentation for your class outlining the methods you used to calculate the land areas.

20. How close are your area calculations to those of your classmates? What might be some of the reasons for your class getting a variety of area values?

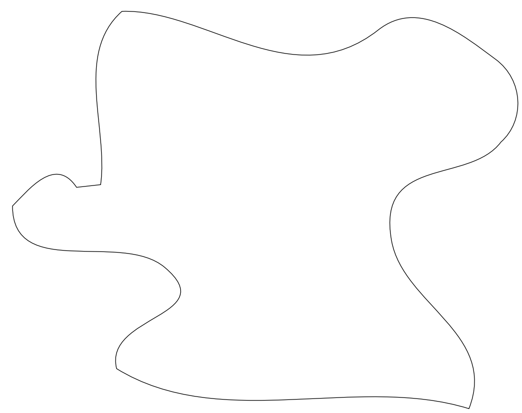
Investigation 4.18

AREAS OF BLOBS AND CIRCLES

Triangles and polygons are convenient shapes to study in geometry, but the truth is, many objects in our world aren't composed of line segments. Circles, egg shapes, and curves of all types are just as common as polygons. How can you find the area of a shape that has curves?

THE BLOB

One way to find the area of a polygon is to divide it into triangles and then find the area of each triangle. But what about a figure that isn't a polygon? For example, how can you estimate the area of this blob? *Can* you find it exactly?



For shapes like this, usually the best you can do is to estimate the area.

FOR DISCUSSION

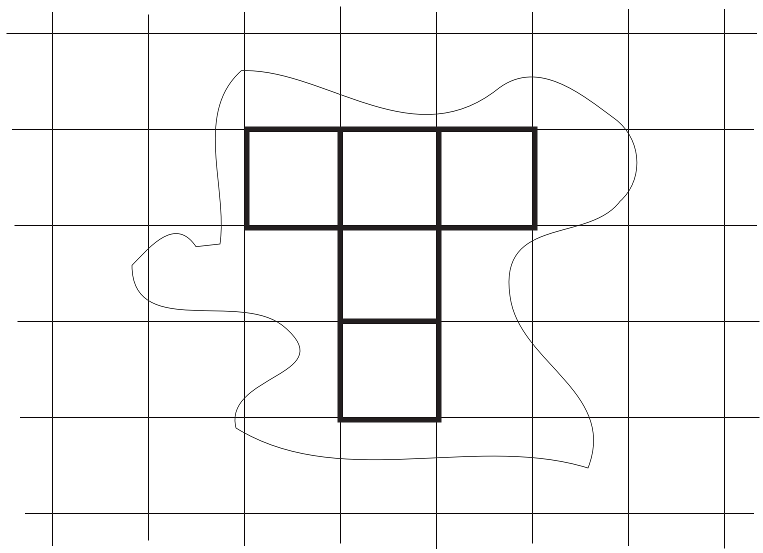
List several ways that you can estimate the area of an irregular shape, such as the "blob" above. Try each of your methods on the blob or some other shape.

An important idea in mathematics is estimating a value by finding upper and lower bounds and then squeezing those bounds closer together.

AREAS OF BLOBS AND CIRCLES (continued)

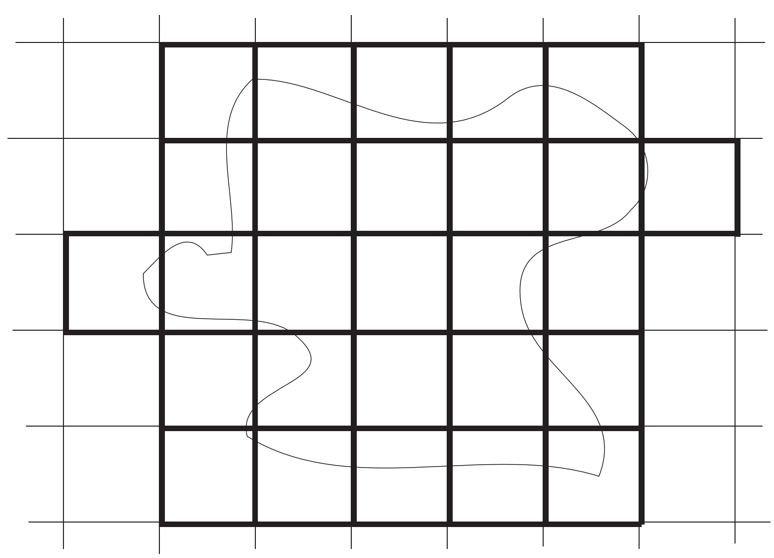
Here's one technique that will let you do this on the blob:

To start, let's put the blob on a piece of graph paper whose squares are $\frac{1}{2}'' \times \frac{1}{2}''$.



Count the number of squares that are completely *inside* the figure. There are 5 of them, each with area $\frac{1}{4}$, so the area of the blob is more than $\frac{5}{4}$ square inches.

Now count all the squares that are either inside *or* touch the blob:

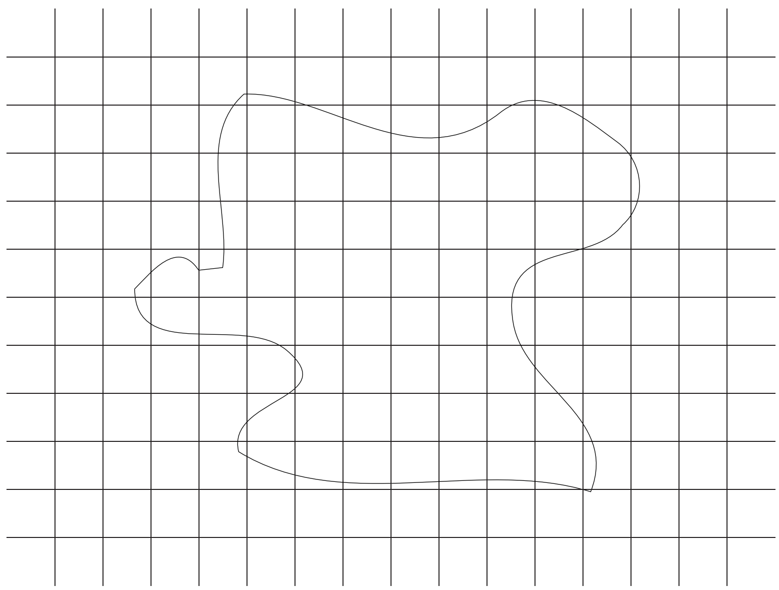


The blob is contained inside 27 squares, so the area of the blob is less than $\frac{27}{4}$ square inches.

So far, we know that the blob has area somewhere between $1\frac{1}{4}$ and $6\frac{3}{4}$ square inches. This is a pretty wide range.

1. How can you improve on this area estimate?

One way to get a closer approximation of the area of the blob is to use graph paper with smaller squares. In this picture, the squares on the graph paper are $\frac{1}{4}'' \times \frac{1}{4}''$:

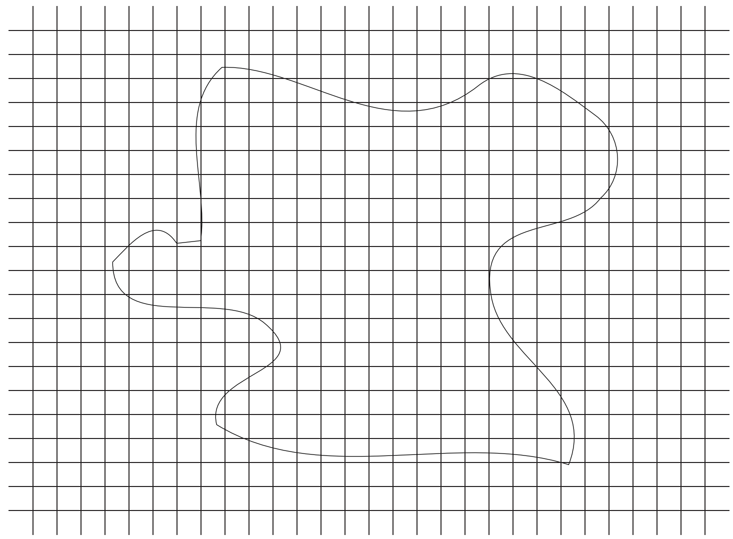


2. What is the area of each little square?

3. Calculating the "inner sum" (the total area of the squares that are completely inside the blob) and the "outer sum" (the total area of the squares that are either inside the blob or touch it) places the area of the blob between two numbers. Are these numbers closer to each other than the first estimate of $1\frac{1}{4}$ and $6\frac{3}{4}$ square inches? Why?

Mathematicians say that you are making a "finer mesh." Window screens have different "mesh sizes." What's the connection?

You probably know what's coming next: Use graph paper with even smaller squares. In this picture, the squares on the graph paper are $\frac{1}{8}'' \times \frac{1}{8}''$:



4. What is the area of each small square?

5. By calculating the inner sum and the outer sum, place the area of the blob between two numbers. Are these numbers closer to each other than the numbers you found in Problem 3?

6. Give an argument to support the claim that as the number of squares per inch gets larger (that is, as the mesh of the graph paper gets *finer*), the difference between the outer sum and the inner sum gets smaller.

WAYS TO THINK ABOUT IT

You now have the basic idea behind how the areas of closed curves (like the blob) are defined. In summary:

1. The region is covered by graph paper, and the inner and outer sums are computed. The mesh is made finer and the process is repeated.

2. This produces a sequence of inner and outer sums. If the difference between these inner and outer sums can be made as small as you want by making the mesh fine enough, then,

3. . . . this means the inner and outer sums get closer and closer to a single number.

4. This single number (the "limit" of the whole process) is the area of the region.

Counting squares can be tedious, but sometimes a shape has symmetry that can make the counting easier by using shortcuts.

7. Draw a circle of radius one foot, and approximate its area using a mesh size of

a. $1''$;

b. $\frac{1}{2}''$;

c. $\frac{1}{4}''$;

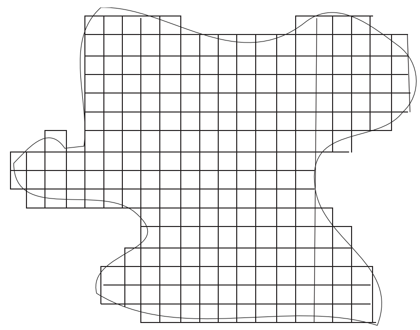
d. $\frac{1}{8}''$.

Use shortcuts wherever possible. Describe any patterns that show up in your estimates.

COMPARING THE AREAS OF BLOBS

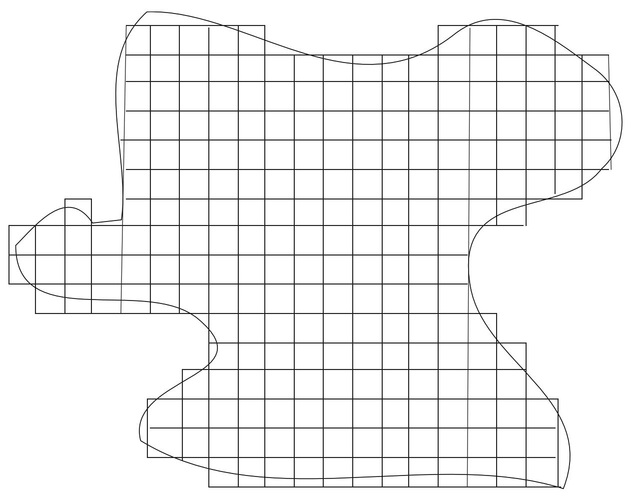
In Investigation 4.17, you found a way to compare the areas of two polygons when one was a scaled copy of the other. If a polygon was scaled by some positive number r, then the ratio of the area of the scaled copy to the original was r^2. Is this true for blobs, too?

8. Imagine that a blob and a grid of squares are drawn onto a big rubber sheet:



The area of these 228 squares gives a pretty good estimate of the blob's area.

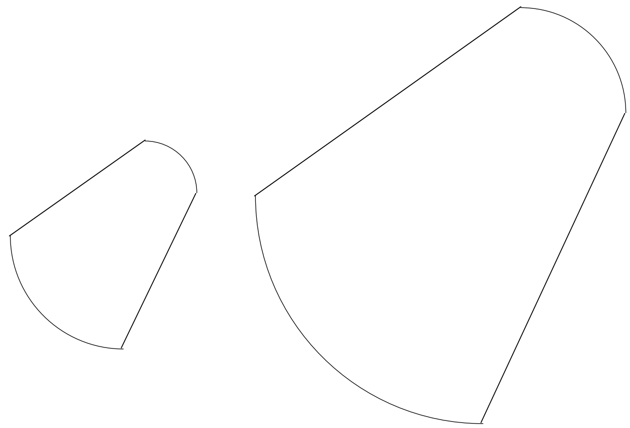
Now imagine that the rubber sheet is stretched uniformly in all directions by a factor of r. This causes the blob and the squares to be scaled by r as well:



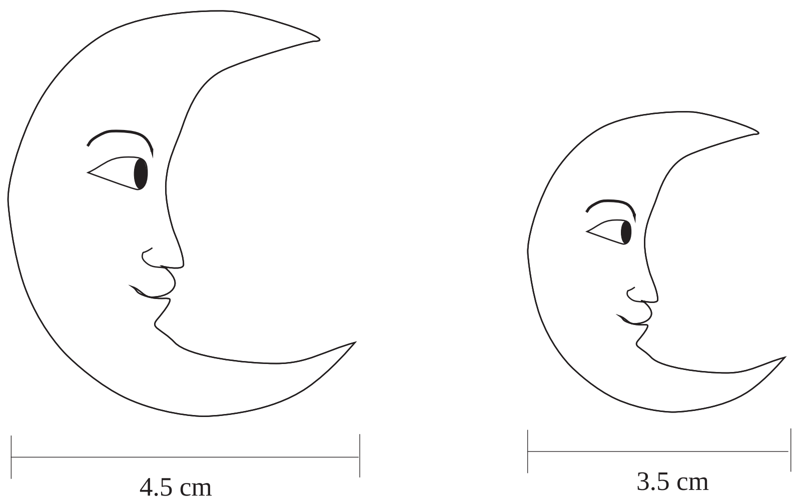
a. The 228 squares still give a good estimate of the area of the blob, but now the area of each square is bigger. By how much has each grown?

b. How has the change in the area of the squares affected the area of the blob?

9. The shape below on the left has an area of 4 square centimeters. The shape on the right is a scaled copy; the scale factor is 2. What is its area?



10. The two crescent moons below are scaled copies of each other. What is the ratio of their areas?



CHECKPOINT

11. Explain in words the method you have used to approximate the area of an irregular shape.

12. **a.** Find the area of a right triangle with sides 3, 4, and 5.

b. Suppose you didn't know the area formula for a triangle. Go through the inner and outer sums process for a 3-4-5 triangle to approximate its area, and see how close you can get to the actual area.

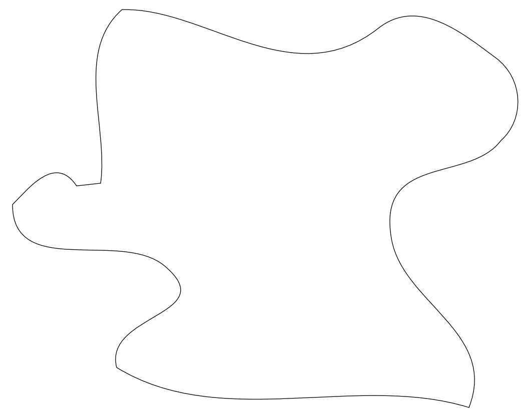
TAKE IT FURTHER

13. Architects, designers, and people who build swimming pools often use a device called a *planimeter* to approximate the areas of irregular shapes. Find out about planimeters and how they work.

Investigation 4.19

PERIMETERS OF BLOBS AND CIRCLES

How can you find the perimeter of a closed curve? That is, given a blob,



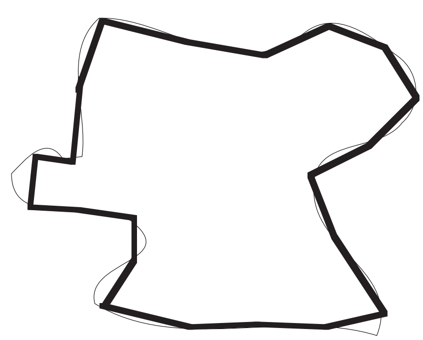
how can you calculate the distance around its edge?

Another way to ask this is "How can you find the *length* of the curve?"

FOR DISCUSSION

Think of several ways to estimate the perimeter of an irregular shape like the blob. Try each of your methods out on the blob or some other shape.

Archimedes used a method for estimating the length of a curved path that is easy to apply. Just "approximate" the curve with line segments and add the lengths of the segments:



1. Approximate the length of the blob or some other curve using this technique.

2. How can you improve your estimate?

3. Many people use this “linear approximation” technique for estimating distance on road maps.

a. Explain how this works.

b. Using a road map and linear approximation, estimate the distance between your hometown and San Francisco. If you live in the San Francisco area, estimate the distance between your hometown and Boston.

4. Tricia has a way to make the linear approximation technique easier and more accurate: She uses what she calls a *regular* approximation for a curve. She picks some length, such as $\frac{1}{4}''$, and sticks with it, marking it off around the curve until she gets too close to the starting point to mark another segment. Then she just multiplies the number of segments by $\frac{1}{4}$ and adds on the last little gap. Try Tricia’s method on a curve.

PERIMETERS OF CIRCLES

You may have learned a formula for the circumference of a circle in an earlier math course. We’ll get to it soon.

Of all curves, perhaps the simplest is the circle. The name given to the length of a circle is one you may already know—the *circumference*. This is just another word that means “perimeter,” but it’s reserved for circles and “round” three-dimensional shapes, such as spheres and cylinders.

WAYS TO THINK ABOUT IT

***Inscribe*: The polygon is contained by the circle, with vertices on the circle. *Circumscribe*: The polygon contains the circle, with sides tangent to the circle.**

The “perimeter” of a circle can be found by a process similar to Tricia’s method in Problem 4 above:

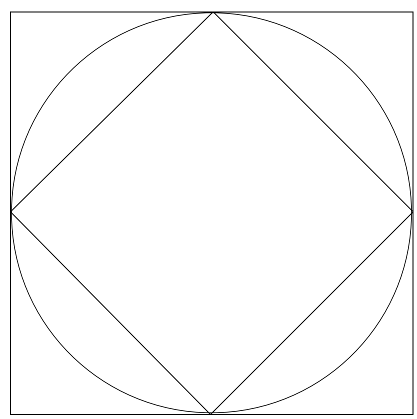
- Inscribe a regular polygon in the circle and circumscribe a regular polygon with the same number of sides around the circle.
- Calculate the perimeter of each polygon.

- Make new regular polygons from the inscribed and circumscribed polygons by doubling the number of sides. Then calculate the perimeters of the new polygons.
- Continue this process. The "inner" and "outer" perimeters will approach a common value, and that number is what is meant by the circle's perimeter.

In Problems 5–7, you will get some practice carrying out the process described above by drawing inscribed and circumscribed polygons for a circle. You may make your own drawings, or you may fill in the table in Problem 8 by taking measurements directly from the drawings provided here.

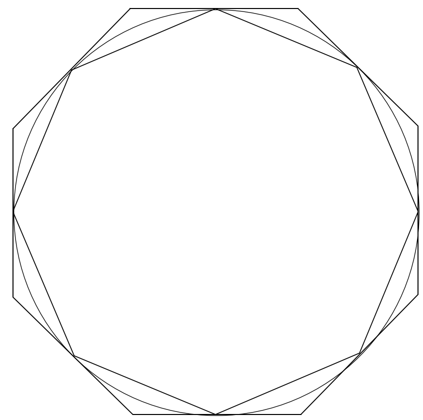
"Draw" means to use either pencil-and-paper drawing tools or geometry software.

5. Draw a circle. Inscribe a square in the circle and circumscribe a square around the circle:



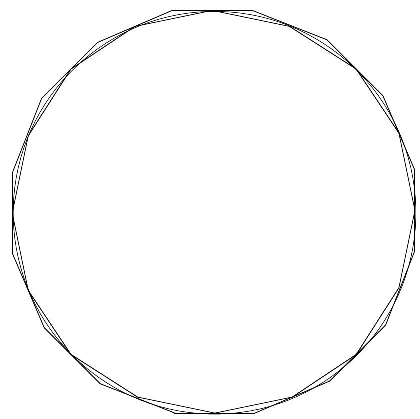
Calculate the perimeters of the two squares, and thus place the circumference of the circle between two numbers.

6. Using the same circle, inscribe a regular octagon in the circle and circumscribe a regular octagon around the circle:



Calculate the perimeter of the two octagons and thus place the circumference of the circle between two numbers.

7. Carry this process one step further with inscribed and circumscribed 16-gons:



8. Use the data from the last three problems to fill in this table:

Number of Sides	Outer Perimeter	Inner Perimeter	Difference
4			
8			
16			

Explain why the difference between the outer and inner perimeters gets smaller as the number of sides gets bigger.

9. Give an approximation for the perimeter of your circle.

CONNECTING AREA AND CIRCUMFERENCE

The idea of approximating a circle with inner and outer polygons leads to a theorem that relates the area of a circle to its circumference:

THEOREM 4.8

The area of a circle is one half its circumference times its radius. In symbols,

$$A = \frac{1}{2}Cr.$$

WAYS TO THINK ABOUT IT

How can we prove this theorem? The formula

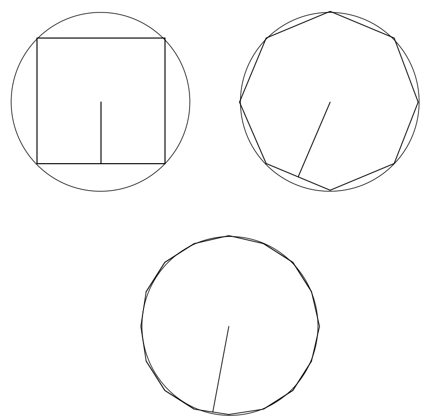
$$A = \frac{1}{2}Cr$$

looks a lot like the formula from Theorem 4.7 in Investigation 4.17:

$$A = \frac{1}{2}Pa.$$

The first formula is about the area of a circle, and the second is about the area of a regular polygon. Throughout this investigation, you've been "approximating" circles with polygons. That's the idea here, too. The idea is to inscribe a sequence of regular polygons in a circle and to study their areas.

Below are three regular polygons inscribed in a circle. The number of polygon sides increases from 4 to 8 to 16. Imagine that these pictures continue on and on with a sequence of regular polygons (with 32, 64, 128, 256, . . . sides) inscribed in this same circle.



As the number of polygon sides increases, it becomes really difficult to distinguish a polygon from the circle and an apothem from a radius.

As the number of sides increases, we'll make the following assumptions:

1. The lengths of the apothems approach the length of the circle's radius.
2. The perimeters of the polygons get closer and closer to the circumference (this is essentially what we mean by circumference).
3. The area of the polygons approaches the area of the circle (this is essentially what we mean by area).

To make this more precise, let's use some notation:

- Let A, C, and r be the area, circumference, and radius of the circle.
- Number the polygons in the sequence 1, 2, 3, ...

- Let the areas of the polygons be $A_1, A_2, A_3, \ldots$, their perimeters be $P_1, P_2, P_3, \ldots$, and their apothems be $a_1, a_2, a_3, \ldots$.

This "arrow" notation means something quite precise in calculus: it means that we can make the difference between the lengths of the apothem and the radius as small as we want by making the *number* of polygon sides big enough.

One more piece of notation: Instead of saying "the lengths of the apothems approach the length of the circle's radius," we will write this as "$a_n \to r$ as n gets larger and larger." Using this shorthand notation, we can rewrite our assumptions:

1. $a_n \to r$
2. $P_n \to C$
3. $A_n \to A$

By Theorem 4.7, for each polygon in the sequence,

$$A_n = \frac{1}{2} P_n a_n.$$

Now, because of our three assumptions, as n gets larger and larger,

$$\frac{1}{2} P_n a_n \to \frac{1}{2} Cr$$

and

$$A_n \to A.$$

You can think of it this way:

$$\begin{array}{ccccc} A_n & = & \frac{1}{2} & P_n & a_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ A & = & \frac{1}{2} & C & r \end{array}$$

To make this proof really airtight, we'd need to fill in several gaps about limits and be more precise about the definitions of area and circumference.

Theorem 4.8 provides a useful connection between a circle's area and its circumference. As you'll see shortly, once you derive the area formula for a circle, you can substitute it into this relationship and obtain a formula for circumference.

TAKE IT FURTHER. . . . .

You can also measure directly from the pictures on page 160.

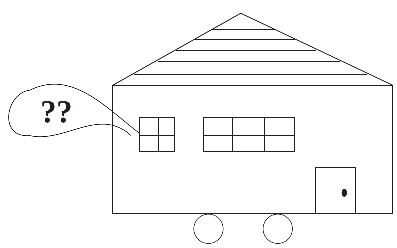
10. Use a ruler and compass or geometry software to fill in this table (all polygons are regular):

Number of Sides	Name	Length of $\frac{\text{perimeter}}{\text{apothem}}$
4	square	
8		
16		

Continue the table for larger numbers of sides.

Do the ratios seem to approach any particular number?

11. The Moriarty sisters, two girls who like to play jokes on their parents, decided one night to pull a prank. They jacked up their family home and slipped two logs under it, each having a circumference of 6 feet.



Then the children started up their mother's bulldozer and began to push the house. They managed to get the logs to roll one revolution before their parents awoke, very dismayed. How far did the Moriarty house move before the sisters were caught in their mischief?

12. Challenge Show that if a regular polygon with n sides and sidelength s is inscribed in a circle of radius 1, then a regular polygon with $2n$ sides inscribed in the same circle has sidelength $\sqrt{2-\sqrt{4-s^2}}$.

Investigation 4.20

CIRCLES AND 3.14159265358979323...

Circles have been discussed quite a bit already. It's time to pull together some ideas.

For starters, all circles are *similar*.

1. Two circles have radii of 12 and 30. Can one of them be scaled to give a congruent copy of the other? Explain.

2. Two circles have radii r and R. Can one of them be scaled to give a congruent copy of the other? Explain.

FOR DISCUSSION

Using ideas from Investigation 4.18, give a plausible argument for the following theorem:

THEOREM 4.9

If a circle is scaled by a positive number r, then the ratio of the area of the scaled copy to the original is r^2.

All the problems in this module are important, but this one is *really* important. Give it some time.

3. In Problem 7 in Investigation 4.18, you approximated the area of a circle with radius one foot. Use that result and the theorem above to find a good approximation for the area of a circle with radius:

a. 2 feet;

b. 5 feet;

c. 6 feet;

d. $\sqrt{3}$ feet;

e. $7\frac{1}{2}$ feet.

4. Give an argument to support this claim:

THEOREM 4.10

If the area of a circle with radius 1 is K, then the area of a circle with radius r is Kr^2.

This theorem says that you can find the area of any circle whatsoever once you know the area of a circle with radius 1. As you've calculated, the value of that area is a bit more than 3. But rather than call it "K," most people call it "pi" and represent it by the Greek letter π:

DEFINITION

π is the value of the area of a circle whose radius is 1.

This is the same π you've used before in math classes. It's usually defined as the ratio of the circumference of a circle to its diameter. The two definitions give you exactly the same number.

Combining Theorem 4.10 with the definition of π gives the result:

THEOREM 4.11

If a circle has radius r, its area is πr^2. In symbols,

$$A = \pi r^2.$$

WAYS TO THINK ABOUT IT

π is a number with a long, fascinating history (see page 170 for some intriguing π facts). But it is, after all, just a number. It's the numerical value of the area of a circle with radius 1. The fact that we give the number such an exotic name shouldn't make you think that it's something

π is the Greek letter for "p."

strange. The area of a circle of radius 1 is a useful number to have around, and rather than saying "area of a circle of radius 1" every time it comes up, people have given it a particular name.

If you ask a person to tell you the value of π, he or she might say 3.14 or $\frac{22}{7}$. While these are indeed *approximations* of π, neither equals π. In fact, π cannot be represented as a ratio of two whole numbers. Its decimal representation is infinite and nonrepeating.

The fraction $\frac{355}{113}$ approximates π accurately to six decimal places.

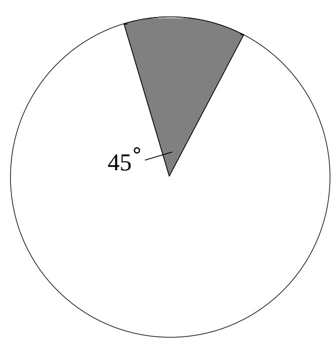
For the purpose of numerical calculations, though, π can be approximated to any degree of accuracy you'd like. One method for doing this is the inner-outer sum method used in Problem 7 in Investigation 4.18: just lay a circle with radius one on graph paper with finer and finer mesh sizes. There are also techniques for approximating π that are ideally suited for computers.

5. Write and Reflect Henri, an inquisitive student, asks,

> "What do you mean 'π is the area of a circle of radius 1?' One what? If you have a circle of radius 1 foot, it can't have the same area as a circle whose radius is 1 inch. This is all nonsense."

Suggest an answer to Henri's question.

Often, people leave the result of calculations about circles in terms of π. That way, anyone who wants a numerical approximation to the result can use whatever approximation for π is handy. For example, if the angle of the "wedge" in this circle is 45° and the radius of the circle is 1,



then the area of the wedge is $\frac{1}{8}$ the area of the circle. Since the area of the circle is π, the area of the wedge is $\frac{\pi}{8}$. Leaving the answer this way conveys more information than saying the area is "about 0.3927."

6. Find the area of a circle if

a. its radius is 10″;

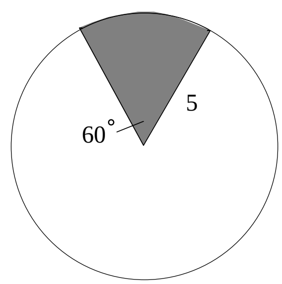
b. its radius is 5 cm;

c. its diameter is 3′;

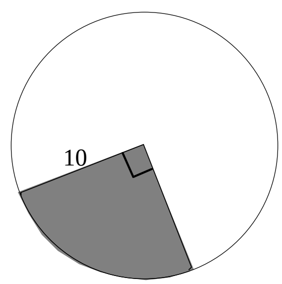
d. it is obtained by scaling a circle of radius 2″ by a factor of 5.

7. Find the area of each shaded region:

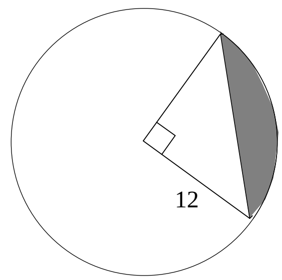
a.



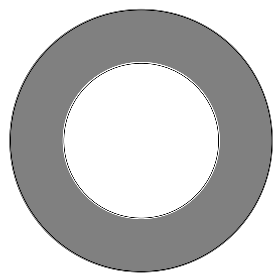
b.



c.



d.



Inner radius is 4, outer radius is 7.

CIRCUMFERENCE

As promised on page 161, you can use Theorem 4.8 to express the circumference of a circle in terms of its radius. Suppose a circle has radius r, circumference C, and area A. By Theorem 4.8,

$$A = \frac{1}{2}Cr.$$

And by Theorem 4.11,

$$A = \pi r^2.$$

So,

$$\frac{1}{2}Cr = \pi r^2.$$

Solving for C, we have a new result:

THEOREM 4.12

The circumference of a circle is twice the product of π and the circle's radius. In symbols,

$$C = 2\pi r.$$

Sometimes, the relation $C = 2\pi r$ is stated in terms of the diameter: $C = \pi d$.

---

8. **a.** Choose the correct answer: The circumference of a circle is approximately five/six/seven times its radius.

b. Choose the correct answer: The circumference of a circle is approximately three/four/five times its diameter.

9. In a circle of radius 2, draw a 60° angle whose vertex is at the center of the circle. The angle cuts the circle into two arcs. How long is each one?

10. True or false? The ratio of a circle's circumference to its diameter is the same for all circles. Explain your answer.

Find a tennis ball canister and check this out!

11. A canister contains three tennis balls. Which distance do you think is longer: the height of the canister or the circumference of the canister?

Guess the answer and then do the calculations to see if your guess was correct.

12. Good 'n' Wormy spaghetti company makes canned spaghetti. The company's cans measure 5″ high and 3″ in diameter. What size piece of paper does the company need to make a label for the outside of its can?

13. The Digi-dial speedometer company makes electronic speedometers for bicycles. The device works by installing a small magnet on a spoke of your wheel and then installing another magnet on the fork of the same wheel so that the two magnets make "contact" every time the wheel makes one revolution. A small computer converts data about the frequency of contacts into miles per hour. When you install one of these speedometers on your bike, you have to set it. One of the numbers you need to know is how far the wheel travels in one revolution. The instructions say to use the "rollout" method: put a chalk mark on the tire where it touches the ground (and mark the ground, too), roll the bike until the mark comes back to the ground, and measure the distance between the chalk marks

with a tape measure. What's an easier way to find the distance for one revolution using what you have learned in this investigation? Try both methods with a bike.

CHECKPOINT

14. The table below gives one piece of information about four different circles. For each circle, find the missing parts:

Radius	Diameter	Area	Circumference
3			
	3		
		3	
			3

TAKE IT FURTHER

15. The Flying Bernoulli Sisters, trapeze artists in the Italian circus, claim to have a way to calculate π. Here's what they do: They calculate two sequences of numbers, n and s_n, and then find $\frac{ns_n}{2}$.

n	s_n	$\frac{ns_n}{2}$
6	1	3
12	0.51763809	3.105828541
24		
48		

Each n is twice the one above it, and each s_n is computed from the previous one by

- squaring the previous one,
- subtracting that answer from 4,
- taking the square root of the result,
- subtracting *that* result from 2, and
- taking the square root of what you get.

In symbols,

$$s_n = \sqrt{2 - \sqrt{4 - (s_{n/2})^2}}.$$

Use a calculator to fill out the Bernoulli table and see if $\frac{ns_n}{2}$ does get close to π. Why does this work? (Hint: See Problem 12 of Investigation 4.19.)

PERSPECTIVE: ALL ABOUT π

The number π has intrigued people for centuries. This essay gives a bit of history and a few notable facts about π.

You can also read a wonderful book devoted just to π: Petr Beckman, *A History of Pi* (Golem Press, 1977).

The numbers given in parentheses represent the years in which the equations were discovered.

π is not the ratio of two integers, but there are lots of ways to approximate it:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \times \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \times \cdots$$

(1579)

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \times \frac{4 \cdot 4}{3 \cdot 5} \times \frac{6 \cdot 6}{5 \cdot 7} \times \cdots$$

(1655)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

(1671)

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

(1734)

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{\ddots}}}}}$$

- The Bible contains the following description:

  "And he made a molten sea, ten cubits from one brim to the other; it was round all about, and his height was five cubits, and a line of thirty cubits did encompass it all around." (Kings 7.23, King James Version).

According to this passage, the molten sea was round with a circumference of 30 cubits and a diameter ("from one brim to the other") of 10 cubits. Calculating the ratio of circumference to diameter gives the biblical approximation of π as $\frac{30}{10} = 3$.

This is known as the method of exhaustion.

- Around 200 B.C., Archimedes found π to be between $3\frac{10}{71}$ and $3\frac{1}{7}$ (about 3.14). To obtain these values, Archimedes calculated the perimeters of 96-sided polygons inscribed in and circumscribed about a circle.
- In the sixteenth century, Ludolph van Ceulen calculated π to 35 decimal places and had the result carved on his tombstone. To this day, Germans still refer to π as *die Ludolphsche Zahl* (the Ludolphine number).
- An English mathematician in 1706 was the first person to use the Greek letter π to represent this number. The symbol was probably intended to stand for the word "periphery."
- The English mathematician Shanks spent 20 years calculating π (without a computer) to 707 decimal places. The results were published in 1873, but sad to say, Shanks made a mistake at the 528th decimal that affected the rest of the digits. Since the mistake was not discovered until 1945, many books perpetuated this mistake by using Shanks' values.
- In 1949, the computer ENIAC took 70 machine hours to calculate π to more than 2000 decimal places. By 1967, a computer had calculated π to 500,000 decimal places in 28 hours. A common way to test a computer's computational reliability is to let it crank out several thousand digits of π and then compare the result to the known value.

A detailed, humorous account of the brothers and π appears in the article "Profiles: The Mountains of Pi" by Richard Preston March 2, 1992 issue of *The New Yorker* (pp. 36–67).

- In 1991, two Russian computer scientists at Columbia University, David and his brother Gregory Chudnovsky, calculated π to more than 2,260,821,336 decimal places. To perform the calculation, the brothers built a supercomputer assembled from mail-order parts and placed it in what used to be the living room of Gregory's apartment. David Chudnovsky says that he and his brother undertook the project because they wanted "to see more of the tail of the dragon."
- Ten decimal places of π would be enough to calculate the circumference of the Earth to within a fraction of an inch if the Earth were a smooth sphere. Thirty-nine decimal places are enough to calculate the circumference of a circle encompassing the known universe with an error no greater than the radius of a hydrogen atom.

How many digits per minute is this?

- The world-record holder for memorizing the most digits of π recited more than 42,000 digits in 17 hours, 21 minutes, which included a break of 4 hours, 15 minutes.

- There are several sites on the Internet devoted to π and its folklore. One site allows you to enter any string of digits, like 92867, and it searches for a matching string of digits in the decimal expansion of π. You might enjoy trying it with your birthdate.

You and your classmates may want to combine the data you collect.

- Do an experiment, either with a computer or by polling people in the halls of your school or at lunch. Get lots of pairs of whole numbers, chosen at random. If you can, get 1000 such pairs. Count the number of pairs that have no common factor (like (5, 8) or (9, 16)). Then take this number and divide it by the total number of pairs. Your answer should be close to

$$\frac{6}{\pi^2} \approx 0.6079.$$

Investigation 4.21

So Many Triangles, So Little Time

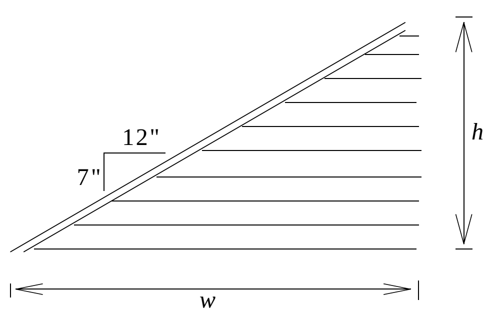
In the final section of this module, you will use similar triangles to find the measurements of unknown triangle lengths. Does this sound familiar? It is, with a new wrinkle added: rather than just solve for triangle lengths, you'll look for patterns and devise shortcuts to make the solution process both faster and more accurate.

Developing these kinds of improvements is immensely important: architects, designers, engineers, and scientists use similar triangles in their work all the time and need reliable, efficient methods to deal with them. By streamlining the solution process, you will discover the basic ideas of the branch of mathematics called *trigonometry*.

All of the problems in this investigation give information about either the lengths or angles of roofs and ask you to find unknown lengths. As you solve the problems, you'll notice there's a lot of repetitive work involved—organizing your measurements and setting up proportions are the kinds of things you need to do over and over again. This investigation offers three proposals for speeding up the solution process. Two of the proposals won't be so fast; the other will be better. See which one you think is best.

Proposal 1: Describing a Roof by Its Pitch

When architects draw a blueprint, they include a little triangle on the roof. This gives the *pitch* of the roof—the number of inches the roof rises for every 12″ of width. In the picture below, the roof rises 7″ for every 12″ of width. Carpenters say this roof has a "7 pitch."



1. For each of the given values of the width, w, calculate the height, h, for a 7-pitch roof.

a. $w = 100$ inches

b. $w = 150$ inches

c. $w = 210$ inches

d. Write a formula that will allow someone to calculate the height h of a 7-pitch roof when given the value of its width, w. Make your formula as simple to use as possible by putting it in the following form:

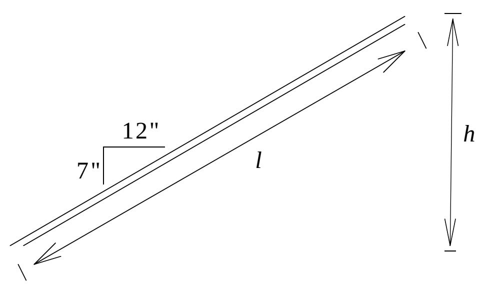
height of 7-pitch roof = *calculation involving w*

2. The roof below has a 7 pitch. For each of the given values of its height, h, calculate its length, l:

a. $h = 130$ inches

b. $h = 170$ inches

c. $h = 250$ inches



d. Write a formula that will allow someone to calculate the length l of a 7-pitch roof when given the value of its height, h. Put your formula in the following form:

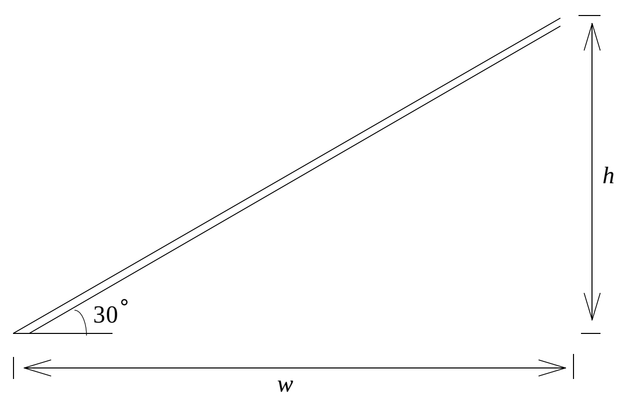
length of 7-pitch roof = *calculation involving h*.

FOR DISCUSSION

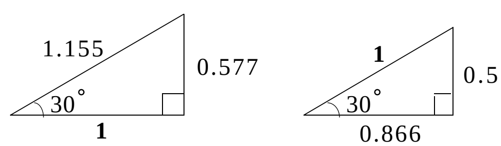
How did you use facts about similar triangles to solve these pitch problems?

PROPOSAL 2: DESCRIBING A ROOF BY ITS UNIT TRIANGLES

Here's a roof that makes a 30° angle with the horizontal:



And here are two right triangles that also have an angle of 30°. Notice that a side of each triangle has a unit length (a length of 1).



(The decimals shown in these figures have been rounded.)

3. Use the triangles above to solve the following roof measurement problems. Decide for each problem which of the two triangles is easiest to use.

a. If $h = 160$ inches, find the length of the roof.

b. If $w = 200$ inches, find h.

c. If $w = 90$ inches, find the length of the roof.

d. If the length of the roof is 250 inches, find h.

4. What is the pitch of a roof with a 30° angle?

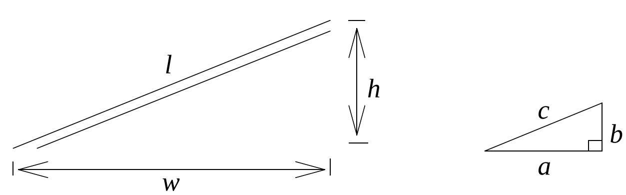
FOR DISCUSSION

How did you use facts about similar triangles to solve these triangle problems?

PROPOSAL 3: DESCRIBING A ROOF BY ITS RATIOS

The picture below shows a roof on the left and a small right triangle with the same pitch on the right. The sides of the small triangle are a, b, and c. Here is some information about the ratios of its sides:

$$\frac{a}{c} = 0.928$$
$$\frac{b}{c} = 0.372$$
$$\frac{b}{a} = 0.401.$$



5. Use the given ratios to solve the following roof measurement problems. For each problem, decide which ratio is easiest to use.

a. If $w = 210$ inches, find the length of the roof.

b. If $h = 300$ inches, find w.

c. If the length of the roof is 140 inches, find h.

FOR DISCUSSION

How did you use facts about similar triangles to solve these ratio problems?

6. **Write and Reflect** Which of the three methods for solving similar triangles (pitch, triangles with one side of unit length, or ratio) do you think is the most useful and allows for the easiest calculations? Why?

7. **Designing Your Own Similarity Chart** Make a "roof-pitch" chart that would be useful to carpenters. The chart should contain pitch angles in one column and then contain enough information in the other columns so that the carpenter can figure out all the lengths that are needed in cutting the roof rafters for that angle.

TAKE IT FURTHER

8. Some hardware stores sell a gadget that helps carpenters measure pitch. If you can find one, figure out how it works and show it to your class.

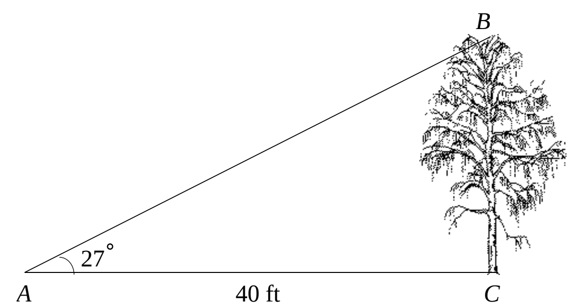
Investigation 4.22

TRIGONOMETRY

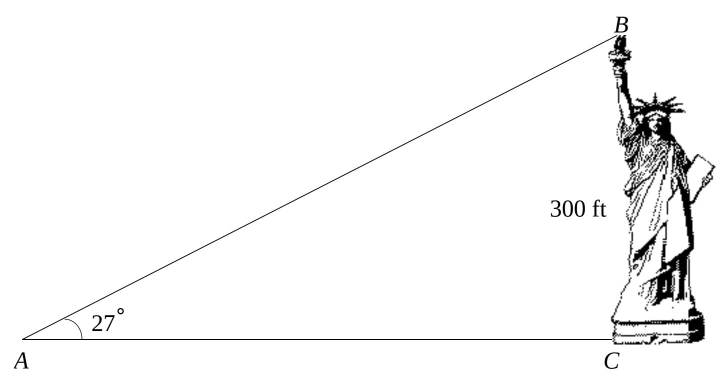
What do these two problems have in common?

- When you're standing at point A, you have to turn your head up by 27° to see the very top of a tree. The distance from you to the tree is 40 feet. How tall is the tree?

This picture isn't exactly right. To simplify the problem, we've overlooked something. What is it?



- When you're in a boat at point A, you have to turn your head up by 27° to see the very top of the Statue of Liberty. If the statue (including its base) is 300 feet tall, how far are you from the bottom of the base?



In both problems, there is a right triangle with an angle measuring 27°. You're given the length of either the triangle's base $\overline{AC}$ or its height $\overline{BC}$, and need to find the other length. Despite the different sizes of the triangles, there is a relationship between AC and BC that's the same in both: their ratio is constant.

1. Explain why the value of $\frac{BC}{AC}$ is the same for both triangles.

2. Measure one of the triangles and calculate this constant value.

3. Use this value to solve both problems.

The problems you've just solved are all examples of trigonometry in action.

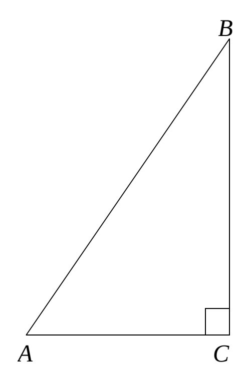
DEFINITION

The word ***trigonometry*** can be split into three parts: *tri* (three) *gon* (side) *metry* (measure). When put together, these words define trigonometry as "the measure of three-sided figures"—in other words, triangles.

You've been measuring triangles throughout this module, so what makes trigonometry so special? Trigonometry tables and trigonometry buttons on calculators give you accurate values of ratios like $\frac{BC}{AC}$ *without* requiring that you draw a triangle and measure its sides every single time.

For easy reference, trigonometry gives names to some of the constant ratios found in a right triangle. Specifically, in right triangle ABC below,

- the *sine* of $\angle A$ is defined as $\frac{BC}{AB}$;
- the *cosine* of $\angle A$ is defined as $\frac{AC}{AB}$;
- the *tangent* of $\angle A$ is defined as $\frac{BC}{AC}$.



There are common shorthand notations for sine, cosine, and tangent:

- "sine of 27°" is abbreviated as "sin 27°";
- "cosine of 27°" is abbreviated as "cos 27°";
- "tangent of 27°" is abbreviated as "tan 27°".

You might need to ask your teacher for some help using a calculator.

You'll find the values of sine, cosine, and tangent for any possible angle by entering them into a calculator or consulting a trigonometry table. For example, if you enter "tan 27°" into a calculator, it gives the value 0.5, to the nearest tenth (try it). This value tells you that for *any* right triangle with $\angle A$ measuring 27°, the ratio $\frac{BC}{AC}$ is 0.5.

4. Rewrite these statements using the language of trigonometry:

These ratios are accurate to two decimal places.

a. In any right triangle ABC with $m\angle A = 40°$, the ratio of the leg opposite $\angle A$ to the hypotenuse is 0.64.

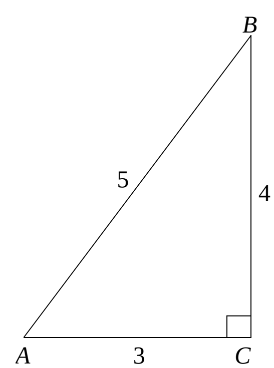
b. In any right triangle DEF with $m\angle E = 70°$, the ratio of the leg adjacent to $\angle E$ to the hypotenuse is 0.34.

c. In any right triangle GHI with $m\angle H = 55°$, the ratio of the leg opposite $\angle H$ to the side adjacent is 1.43.

5. **a.** Find the values of sin A, cos A, and tan A for the triangle below.

b. Find the values of sin B, cos B, and tan B.

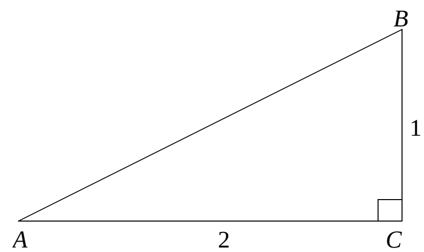
c. Which of your answers from a and b are the same? Can you explain why?



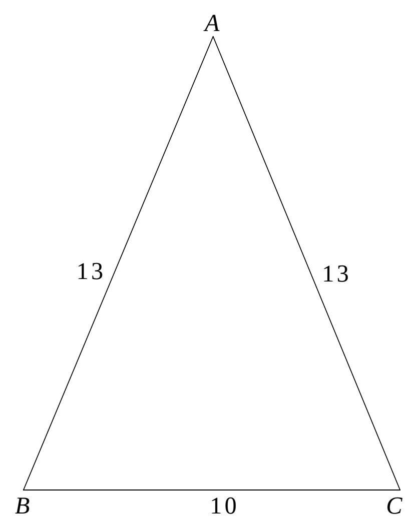
You will need to use the Pythagorean Theorem for this problem.

6. a. Find the values of $\sin A$, $\cos A$, and $\tan A$ for the triangle below.

b. Find the values of $\sin B$, $\cos B$, and $\tan B$.



7. Find the values of $\sin B$ and $\cos B$ for isosceles triangle ABC:



Drawing a sketch will help.

8. $\triangle RST$ is a right triangle with right angle at S. If $\tan R = \frac{2}{3}$, then find the values of $\sin R$ and $\cos R$.

9. $\triangle JKL$ is a right triangle with right angle at K. If side $\overline{JK}$ is three times the length of $\overline{KL}$, find the sine, cosine, and tangent values for $\angle J$ and $\angle L$.

10. The piece of paper drawn below originally showed a complete right triangle, $\triangle ABC$, with right angle at C. The paper was ripped, though, so all you can see now is $\angle A$ (which measures 40°). Find as many of these values as you can using a calculator. Some might not be possible.

a. $\frac{BC}{AC}$

b. $AC + BC$

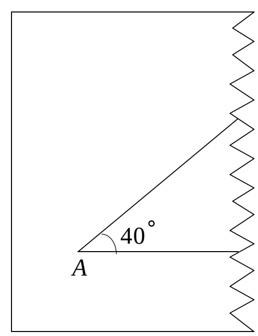
c. $\frac{BC}{AB}$

d. $AB \times AC$

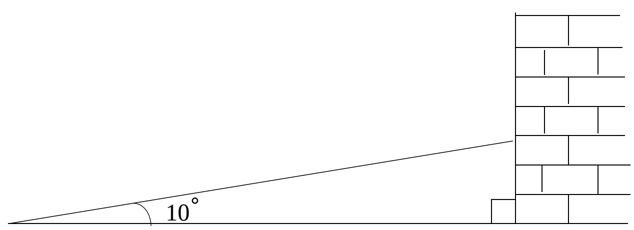
e. $\frac{AC}{AB}$

f. $AB - BC$

g. $\frac{AC}{BC}$



11. A handicapped-access ramp slopes at a 10° angle:



Use a calculator to solve this problem and the next one.

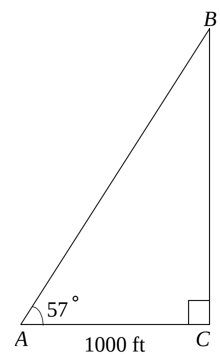
a. If the ramp meets the ground 25 feet from the base of the building, how long is the ramp?

b. Most public buildings were built before handicapped-access ramps became widespread, so when it came time to design the ramps, the doors of buildings

were already in place. Suppose a particular building has a door 2 feet off the ground. How long must a ramp be to reach the door if the ramp is to make a 10° angle with the ground?

12. An airplane takes off and flies 10,000 feet in a straight line, making a 25° angle with the ground. How high above the ground does the airplane rise?

13. To the nearest tenth, the value of $\tan 57^\circ$ is 1.5. To the nearest thousandth, it's 1.540. Solve for the length of $\overline{BC}$ in the triangle below twice, using each of these values. By how much do your two answers differ?

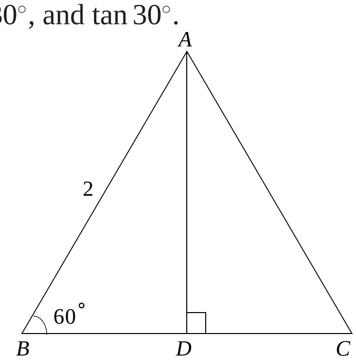


14. It's possible to find the exact values of the sine, cosine, and tangent for a few special angles by using what you know about equilateral and isosceles triangles:

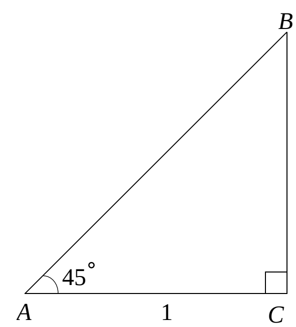
a. $\triangle ABC$ below is an equilateral triangle with altitude $\overline{AD}$ drawn from vertex A. Find the lengths of $\overline{BD}$ and $\overline{AD}$. Use these lengths to find the exact values of

i. $\sin 60^\circ$, $\cos 60^\circ$, and $\tan 60^\circ$;

ii. $\sin 30^\circ$, $\cos 30^\circ$, and $\tan 30^\circ$.



b. $\triangle ABC$ below is an isosceles right triangle with a leg of length 1. Use this triangle to find the exact values of $\sin 45^\circ$, $\cos 45^\circ$, and $\tan 45^\circ$.



15. Is there an angle between 0° and 90° for which the sine of that angle equals 1.5? Explain.

CHECKPOINT

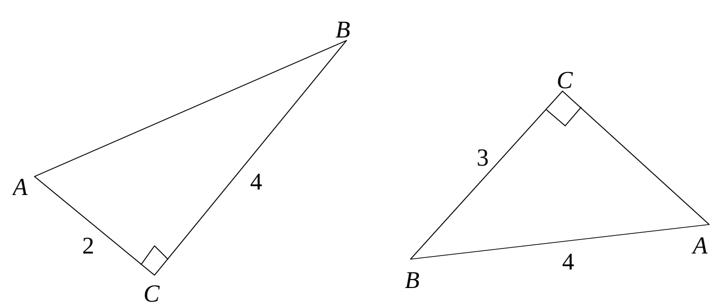
16. Define the following terms:

a. sine of an angle;

b. cosine of an angle;

c. tangent of an angle.

17. For each of the right triangles below, find the exact values for sine, cosine, and tangent of $\angle A$ and $\angle B$.

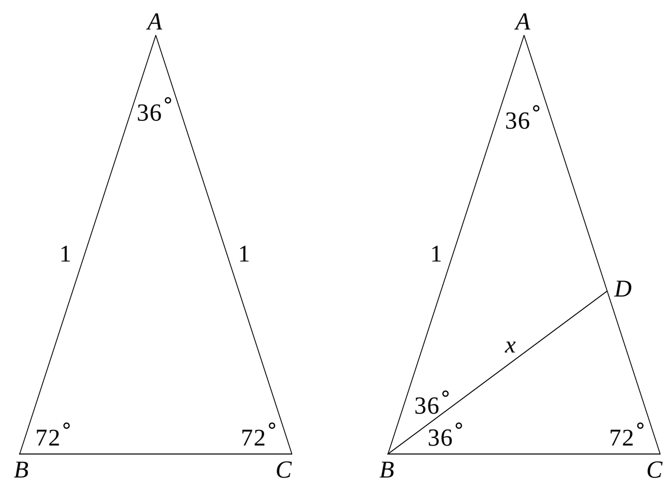


TAKE IT FURTHER

18. Here's a clever way to find the exact value of $\cos 72^\circ$:

The isosceles triangle ABC on the left has base angles measuring 72°, and $AB = AC = 1$.

The same triangle is shown on the right with segment $\overline{BD}$ bisecting $\angle B$. Segment $\overline{BD}$'s unknown length is labeled x.



a. Find the lengths of $\overline{BC}$, $\overline{AD}$, and $\overline{DC}$ in terms of x.

b. Explain why $\triangle ABC \sim \triangle BCD$.

c. Set up a proportion involving the lengths AB, BC, and CD.

d. Use this proportion and the quadratic formula to solve for x.

e. Divide $\triangle ABC$ into two right triangles by drawing its altitude from point A.

f. Use either of these right triangles and the value of x to find $\cos 72^\circ$.

Now find $\cos 72^\circ$ on a calculator. Compare the value the calculator gives to the exact value you have found.

19. By dividing $\triangle ABD$ above into two right triangles, find $\cos 36^\circ$. As in Problem 18, compare the value a calculator gives for $\cos 36^\circ$ to this exact value.

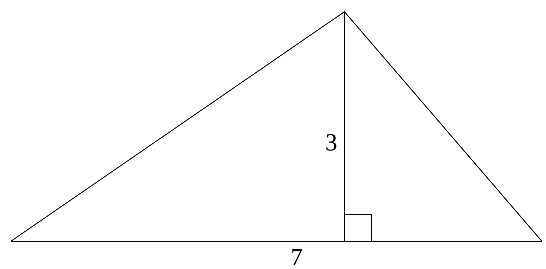
Investigation 4.23

AN AREA FORMULA FOR TRIANGLES

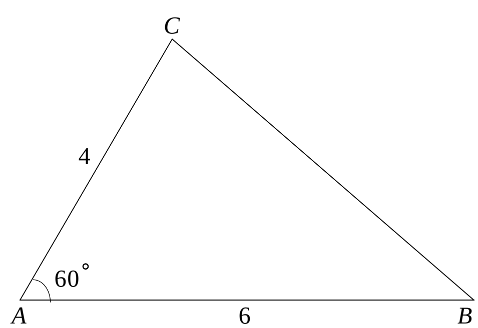
You already know the standard area formula for a triangle:

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}).$$

Using this formula, you know that the area of the triangle below is $\frac{1}{2}(7)(3) = 10.5$.



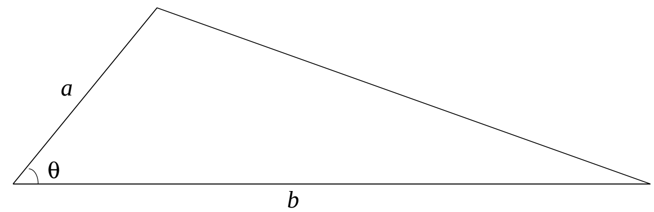
But how can you find the height of a triangle if it's not given? For example, look at this triangle:



1. Find the area of $\triangle ABC$ by first calculating the length of the altitude from C. Trigonometry can help you here.

θ **(the Greek letter "theta") is a variable often used to represent angles or angle measures.**

2. **a.** Find the area of this triangle in terms of a, b, and θ:



b. Use the area formula from part a to show that if the triangle is scaled by a factor of r, then its area is multiplied by r^2.

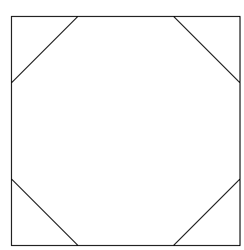
3. Find the area of parallelogram $ABCD$:

TAKE IT FURTHER

4. Find the area of a regular octagon that has a sidelength of 4 inches.

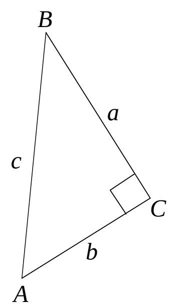
Hint: Divide the octagon into triangles and find the area of each one.

You might also consider this picture that shows the octagon sitting in a square. If you can find the area of the square and the four triangles, how will this help?



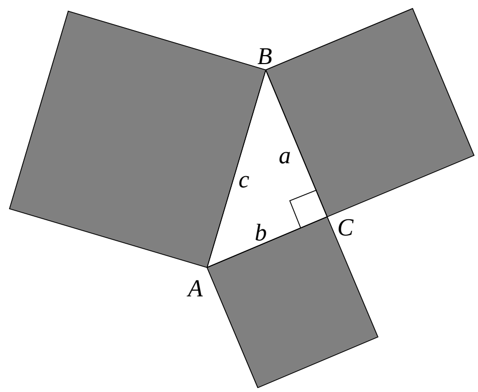
Investigation 4.24

Extending the Pythagorean Theorem



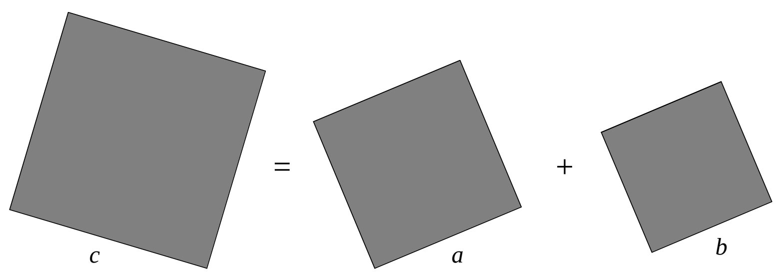
For any right triangle, the Pythagorean Theorem gives you a way to relate the lengths of the three sides. In right triangle *ABC* shown in the margin, $a^2 + b^2 = c^2$.

Geometrically speaking, you can think of a^2, b^2, and c^2 as the areas of squares drawn on the three sides of $\triangle ABC$.



The Connected Geometry module *The Cutting Edge* gives several geometric dissection proofs of this theorem.

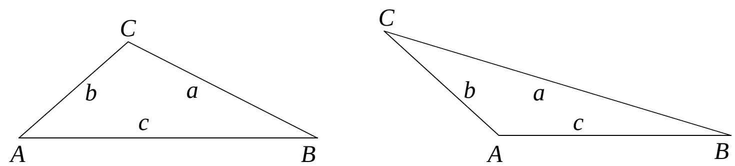
Since $a^2 + b^2 = c^2$, the area of the square built on the hypotenuse is equal to the sum of the areas of the squares sitting on the other two sides:



What happens, though, for triangles with *no* right angle? Does something similar to the Pythagorean Theorem still apply?

1. Use geometry software to draw an arbitrary triangle. Construct squares on its three sides and measure their areas. Add the areas of two squares to see if their sum is the third. Try different combinations of squares. Does the Pythagorean relationship apply? Experiment with different triangles by moving the vertices or sides of your original triangle.

2. Here are two triangles with vertices labeled A, B, and C. $\angle C$ is obtuse in one triangle and acute in the other:

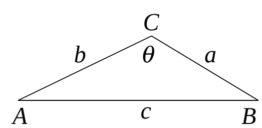


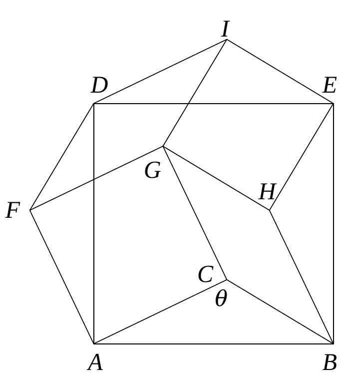
Based on your work from Problem 1, can you predict for each triangle whether $a^2 + b^2$ will be less than, greater than, or equal to c^2?

There is a way to extend the Pythagorean Theorem to a more general theorem that applies to *all* triangles, with or without right angles. The geometric recipe below shows you how.

A Recipe for Extending the Pythagorean Theorem

1. Using geometry software, draw a triangle ABC with $\angle C$ greater than 90°. In the picture, the measure of $\angle C$ is labeled θ.





2. Construct a square on side $\overline{AB}$ of your triangle so that the square covers the triangle. Compared to the squares in the Pythagorean Theorem, this one faces the opposite way.

3. Construct a square on side $\overline{AC}$ of your triangle.

4. Construct a parallelogram with sides $\overline{CB}$ and $\overline{CG}$.

5. Construct a square on side $\overline{GH}$ of your parallelogram.

6. Finally, construct a parallelogram with sides $\overline{GF}$ and $\overline{GI}$.

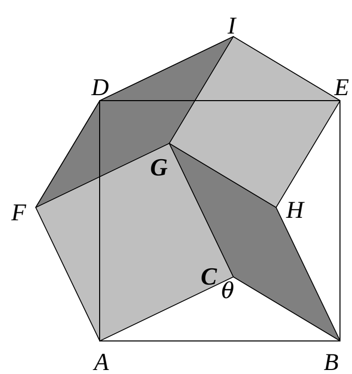
As you manipulate the figure, you might get the sensation that you're moving an actual physical model composed of hinges at the vertices and taut rubber bands on the sides.

3. **Write and Reflect** Now that your construction is complete, drag vertex C of $\triangle ABC$ and watch what happens. Which features of the construction change and which stay the same? Make note of any regions that appear to be congruent to each other.

COMPARING AREAS

Your triangle ABC has five quadrilaterals to accompany it. Specifically, there's a "big" square $ADEB$, two "small" squares $AFGC$ and $GIEH$, and the two parallelograms $CGHB$ and $FDIG$.

To help keep track of the two small squares and the two parallelograms, use the software to shade each a different color or darkness.



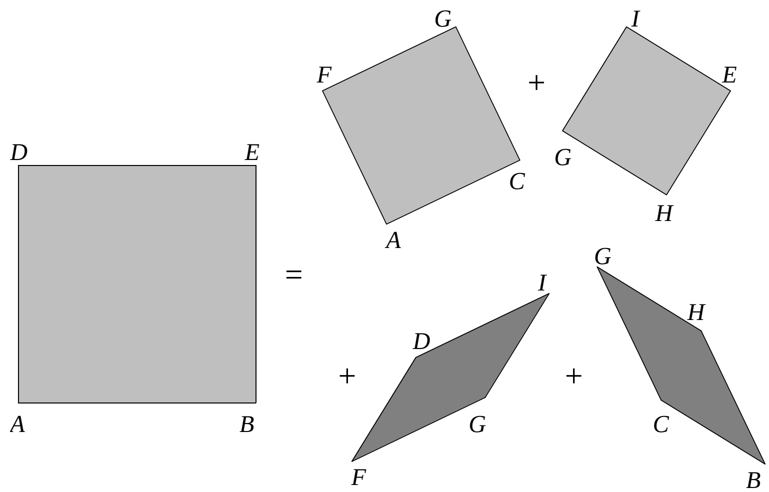
4. Experiment with different locations of point C (keeping θ greater than $90°$), and observe how the following two regions compare in area:

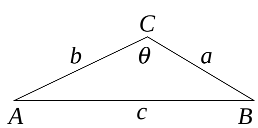
- **Region 1:** the space occupied by the two small squares and the two parallelograms.
- **Region 2:** the space occupied by the large square $ADEB$.

Notice that much of the two small squares and two parallelograms overlap the large square. You have to get only those pieces that lie outside the large square to fit snugly inside it.

5. Devise a way to cut up and rearrange the two small squares and two parallelograms (Region 1) so that they fit exactly within the large square (Region 2). Write down your method or draw a picture to illustrate it.

6. Explain what this visual equality is saying:





7. What is the area of each of the three squares in the picture above? Your answers should be in terms of just a, b, and c. Remember that in $\triangle ABC$, $BC = a$, $AC = b$, and $AB = c$.

8. What are the sidelengths and angle measurements of the two parallelograms in the picture? Your answers should be in terms of just a, b, c, and θ.

9. **Write and Reflect** Explain how the theorem on the next page applies to $\triangle ABC$. To make the theorem complete, include the sidelengths and angle measurements of the parallelograms.

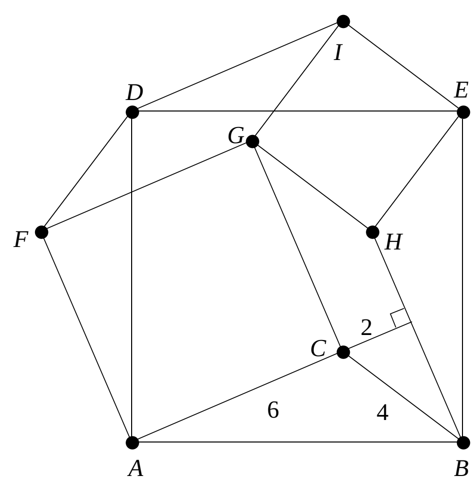
THEOREM 4.13

For $\triangle ABC$ with sides a, b, and c, and θ obtuse,

$$c^2 = a^2 + b^2 + 2(\underline{\qquad\qquad}).$$

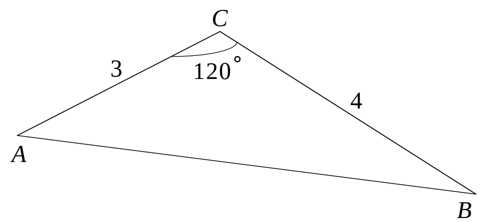
PUTTING THE PYTHAGOREAN EXTENSION TO WORK

10. Use Theorem 4.13 to solve for AB in the figure below:

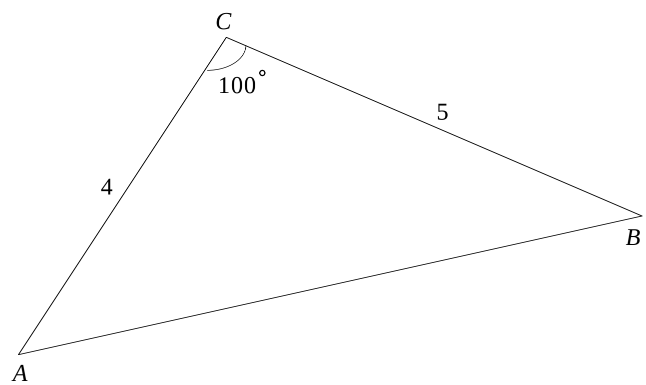


11. Use Theorem 4.13 to solve for the length of side $\overline{AB}$ in each of these two triangles:

a.



b.



CHECKPOINT

12. Why is the extension of the Pythagorean Theorem more powerful than the Pythagorean Theorem itself? What new kinds of problems does it allow you to solve?

TAKE IT FURTHER

13. Use software to see what happens to the two parallelograms in your picture when θ equals 90°. Can you use this setup to give a geometric dissection proof of the Pythagorean Theorem?

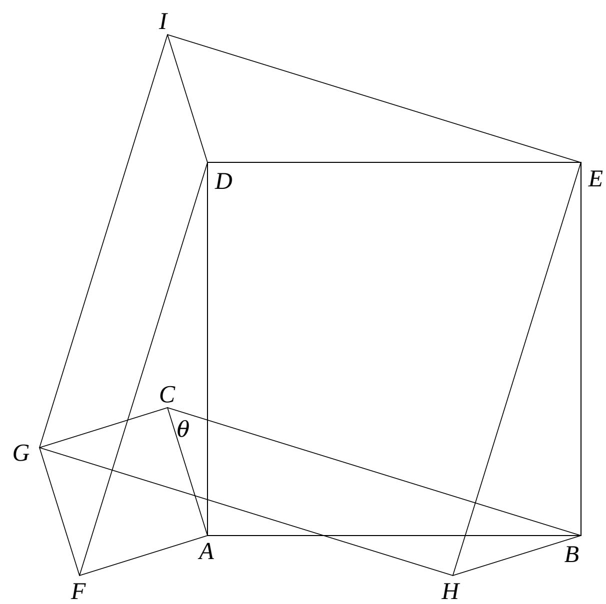
14. Move point C so that it lies anywhere along $\overline{AB}$. What happens to $\triangle ABC$? Write down the algebraic identity represented by this new picture.

15. The extension of the Pythagorean Theorem works when θ is greater than or equal to 90°. How about triangles in which θ is *less* than 90°? For these triangles, the areas of parallelograms *FDIG* and *CGHB* get subtracted from, rather than added to, the areas of the two small squares:

$$\text{Area}(ADEB) = \text{Area}(AFGC) + \text{Area}(GIEH) - \text{Area}(FDIG) - \text{Area}(CGHB).$$

Again, a dissection argument will help you prove this:

Move point C so that the measure of θ is now acute. Chances are, your picture has become very messy! Rather than trying to interpret all of the shading the picture already contains, it's easier to start from scratch and add and subtract the shaded regions one at a time.



Copy the figure above and use a pencil to shade in the areas of squares *AFGC* and *GIEH*. You'll be shading some regions twice, so you might want to write the number "2" in these places to keep track of them. Then, use

your pencil's eraser to remove/subtract the areas of the two parallelograms *FDIG* and *CGHB*. If you're subtracting a region that's not already shaded, you might want to outline it and indicate that the region must be removed. When you're done, figure out how to fit everything that's left into the square *ADEB*.

MODULE OVERVIEW AND PLANNING GUIDES

ABOUT THE MODULE

This module focuses on three related overarching themes: scale drawing, similarity, and trigonometry.

"Maps, Blueprints, and Scale Factors" (Investigations 4.1–4.7) introduces the concept of scaling. Students read maps and blueprints, using the provided scales to calculate distances and lengths. They examine enlargements and reductions of a horse-skeleton picture—some accurate and some not—to formulate their own tests for deciding what makes a well-scaled copy. Finally, they apply this knowledge to pairs of triangles and pairs of polygons, again developing methods to recognize scaled copies.

"Constructing Enlargements and Reductions" (Investigations 4.8–4.13) develops two different techniques for creating scale drawings, both related to the notion of *dilation*. Students use these methods to draw their own scaled copies, and then prove that the methods indeed work. The proofs establish two important triangle similarity results: the Side-Splitting Theorem and the Parallel Theorem.

"Similarity and Its Applications" (Investigations 4.14–4.16) begins by proving some classic triangle similarity tests, including AA, SAS, and SSS. Students use these tests to explore a variety of simple, yet accurate, methods for determining unknown distances and heights. Aside from standard distance calculations, this section of the module also provides a range of other similarity applications. A "segment splitter" activity asks students to develop several methods for dividing a segment into any number of congruent parts by using only a straightedge and a lined sheet of notebook paper. The investigation concludes with a historical profile of Sarah Marks, a science researcher from the turn of the century who patented her own segment splitter. A copy of her patent is included, and students are asked to explain how Marks' device works as well as relate it to any of the methods they have devised.

Another similarity application challenges students to use geometry software to build a rectangle that both stays a rectangle when any of its parts is moved and maintains its area. A follow-up activity connects this work to the concept of the geometric mean.

"Areas, Circles, and π" (Investigations 4.17–4.20) looks at how the area and perimeter of objects are affected by scaling. It discusses how one might determine the area and perimeter of objects with curves and develops proofs for the area and circumference formulas of a circle (along with an historical overview of π). Throughout this section, students receive an intuitive introduction to the notion of limits.

"An Introduction to Trigonometry" (Investigations 4.21–4.24) encourages students to develop their own methods for finding unknown sidelengths of triangles well before it introduces the names of trigonometric ratios. Students explore three different methods for finding the measurements of a sloped roof. They ultimately decide for themselves what kind of information is the most useful to provide when looking for the lengths of unknown triangle sides.

The trigonometry section concludes with an investigation that relates to the Pythagorean Theorem work students may have done in the module *The Cutting Edge*. Using a dissection argument similar to the Pythagorean Theorem proofs in *The Cutting Edge*, students develop a geometric way to relate the three sides of *any* triangle to each other, not just those with right angles. This geometric approach is, in fact, equivalent to the algebraic statement of the Law of Cosines.

MAIN TIMELINE PLANNING CHART

Below you will find a brief description of each investigation in this module, the specific content it covers, and an estimated teaching time. Teaching the complete module would require 9 to 12 weeks of class time.

Investigation	Description	Key Content	Pacing
4.1 Introduction to Maps and Blueprints	Students use a map of Seattle to compute distances between various downtown locations. Students also use a blueprint to compute various measurements of a house.	• scale of a map • blueprints • scaling	2 days
4.2 What Is a Scale Factor?	This investigation defines *scale factor* and asks students to compute the amount by which various figures have been scaled.	• defining *scale factor* • area, volume	2 days
4.3 Working with Directions: A Logo Activity	Students use the computer program Logo to draw scaled figures and explore self-similarity/fractals.	• Logo • self-similarity	2–3 days

Module Overview and Planning Guides

(continued)

Investigation	Description	Key Content	Pacing
4.4 What Is a Well-Scaled Drawing?	Students examine enlargements and reductions of a horse-skeleton picture—some accurate and some not—to formulate their own tests for deciding what makes a well-scaled copy.	• properties of scaling	1–2 days
4.5 Testing for Scale	Students develop tests for checking whether pairs of rectangles, triangles, and polygons are scaled copies.	• corresponding sides/angles • defining *proportional*	3 days
4.6 The Many Faces of Scaling	The features of scaled copies are reviewed within a more artistic setting.	• scaling review	1 day
4.7 Rectangle Diagonals	Students repeatedly fold and tear a rectangle into a collection of smaller rectangles, and then develop a test to check which rectangles are scaled copies of each other.	• rectangles • scaling tests	1–2 days
4.8 Light and Shadows: Projected Images	Students explore scaled pictures in the context of movie projectors and shadows.	• shadows • projection	1 day
4.9 Curved or Straight? Just Dilate!	A dilation method for scaling both polygons and curved figures is introduced.	• defining *dilation*	2 days
4.10 Ratio and Parallel Methods	Two different techniques for scaling a picture using the concept of dilation are introduced.	• ratio and parallel methods • dilation	2–3 days
4.11 Nested Triangles: Building Dilated Polygons	The Parallel and Side-Splitting Theorems for triangles are introduced.	• parallel theorem • side-splitting theorem	2–3 days
4.12 Side-Splitting and Parallel Theorems	Several proofs of the theorems introduced in the previous investigation are provided. Students solve several problems that use the Parallel and Side-Splitting Theorems to prove results about dilations. Students prove that the figure formed by connecting the midpoints of an arbitrary quadrilateral is a parallelogram and then solve related problems.	• proof • comparing areas • parallelograms	6 days

Investigation	Description	Key Content	Pacing
4.13 Historical Perspective: Parallel Lines	Students prove several results from Euclid's *The Elements*. Eratosthenes' method for approximating the Earth's circumference is also discussed.	• Euclid • Eratosthenes	2–3 days
4.14 Defining Similarity	The term "similarity" is defined, and the notation used to describe similar figures is introduced.	• defining *similarity* • similarity notation	2 days
4.15 Similar Triangles	Students prove the AA, SAS, and SSS triangle similarity theorems.	• AA, SAS, SSS	3–4 days
4.16 Using Similarity			
Calculating Distances and Heights	Students find the values of unknown distances and heights by using similar triangles.	• "real-world" applications	2 days
Segment Splitters	Several experiments allow students to split a segment into any number of congruent pieces without taking a single measurement.	• segment splitting	3 days
A Constant-Area Rectangle	Students use the Power-of-a-Point Theorem to create a *constant-area rectangle* with geometry software.	• power of a point • constant area	3 days
The Geometric Mean	The construction from the constant-area rectangle is used to introduce the concept of the geometric mean.	• defining *geometric mean*	3 days
No Measuring, Please!	Using results from previous investigations, students are challenged to construct lengths in a purely geometric fashion without taking any measurements.	• constructing lengths	2 days
4.17 Areas of Similar Polygons	The effects of scaling on the areas of rectangles, triangles, and general polygons are explored.	• area • apothem	3–4 days
4.18 Areas of Blobs and Circles	The effects of scaling on closed curves and circles are explored.	• area, closed curves • limits	3–4 days

Investigation	Description	Key Content	Pacing
4.19 Perimeters of Blobs and Circles	Techniques for finding the perimeters of curves and approximating the circumference of circles are introduced. A formula that relates the area of a circle to its circumference is developed.	• perimeter • circles, circumference	3–4 days
4.20 Circles and 3.1415926535897932...	The area and circumference formulas for circles are derived and an historical introduction to π is provided.	• circle area/circumference • pi	3–4 days
4.21 So Many Triangles, So Little Time	As an introduction to the methods of trigonometry, students explore three different ways to calculate the lengths of roofs.	• roof "pitch" • right triangles	2–3 days
4.22 Trigonometry	Trigonometric vocabulary is introduced in the context of right-triangle problems.	• trigonometry • sine, cosine, tangent	3–4 days
4.23 An Area Formula for Triangles	Students use trigonometry to derive the triangle area formula $A = \frac{1}{2}ab\sin\theta$.	• triangle area • trigonometry	1–2 days
4.24 Extending the Pythagorean Theorem	Students explore a method for finding the length of unknown triangle sides by extending the results of the Pythagorean Theorem. Although the method is never formally named, it is, in fact, a version of the Law of Cosines.	• Pythagorean Theorem • dissection proof • Law of Cosines	3–4 days

ALTERNATE TIMELINES

It is very likely that you will not have time to complete this entire module with your class. We provide more investigations and problems than you will use because we want to give you the maximum level of flexibility in adapting the material to your needs and interests. Here are some suggestions for selecting investigations.

Scaling
- 4.1 (either section) (1 day)
- 4.2 (2 days)
- 4.4 (2 days)
- 4.5 (3 days)
- 4.6 (1 day)

An Introduction to the Ideas of Scaling (2 weeks)

This option provides your students with the basic ideas behind scaling, including map and/or blueprint reading, determining the features of well-scaled drawings, and methods to test for scaled copies.

The Dilation Method for Creating Similar Figures (3 weeks)

The section "Constructing Enlargements and Reductions" (Investigations 4.8–4.13) introduces two dilation methods for drawing similar figures and then ties the methods to the Side-Splitting and Parallel Theorems for triangles.

Proof-Focused
- 4.10 (2 days)
- 4.11 (2 days)
- 4.12 (6 days)
- 4.13 (2 days)
- 4.14 (2 days)
- 4.15 (3 days)
- 4.16 (choose 2) (6 days)

Focus on Proof (4 to 5 weeks)

This timeline is for those who want to take the fullest advantage of the module's offering on proof. You may want to add some of the introductory investigations, investigations on area from the section "Areas, Circles, and π" and investigations on trigonometry from the section "An Introduction to Trigonometry." The focus in the timeline presented here is scaling, parallel line theorems, and proof of the similar triangle tests.

Applications
- 4.7 (1–2 days)
- 4.12 (2 days)
- 4.16 (all five investigations) (2 to 3 days each)

Applications of Similarity (3 to 4 weeks)

There are numerous investigations in this module that provide applications of similarity. If your class is already familiar with similarity, you can select from among these.

Trigonometry (2 weeks)

If you would like to introduce your students to the fundamental concepts of right-angle trigonometry, the section "An Introduction to Trigonometry" (Investigations 4.21–4.24) of this module can be used by itself.

The Effects of Scaling on Area and Perimeter (2 to 3 weeks)

The section "Area, Circles, and π" (Investigations 4.17–4.20) is a self-contained unit that asks the following central question: How does scaling affect the areas and perimeters of polygons and figures with curves? To answer this question for curved figures, the Student Module introduces students to some fundamental ideas from calculus, including approximations and limits. The material is treated in a concrete manner, using the squares on graph paper as a means to approximate area. The area and circumference of a circle are taken as a special case, with the formulas $A = \pi r^2$ and $C = 2\pi r$ developed from scratch.

ASSESSMENT PLANNING

Throughout the entire module, we recommend that students keep a notebook containing

- daily homework and other written assignments
- a list of vocabulary, definitions, and theorems that emerge during classwork and homework

What to Assess

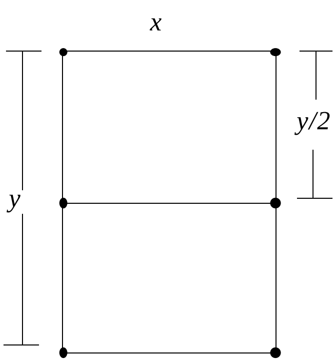
- The student understands that "scale factor" measures the change in linear dimension when a picture is scaled.
- The student can approximate the scale factor between two pictures by measuring.
- The student can use a given scale factor to intepret a map or blueprint.
- The student can decide if two figures are well-scaled copies of each other (testing if corresponding linear measurements are in the same proportions.)
- The student can articulate and carry out at least one test for triangle similarity.
- The student can describe and carry out the "diagonal test" for polygon similarity.
- The student can describe and use two methods for constructing enlargements or reductions of shapes.
- The student can apply and explain the ratio and parallel methods and the Parallel and Side-Splitting Theorems.
- If you cover Investigation 4.12, students should be able to outline proofs of the theorems based on area. You can give students the option of presenting or writing up the proof, but you should require some kind of write-up of the presentation as well.
- The student can define similarity in terms of scaling or dilation. The student can draw conclusions about angles and sides of similar triangles.
- The student can articulate and use tests for triangle similarity: AA, SAS, SSS. You may want to add proofs and applications.

- The student can state and apply the fact that area of scaled figures is related by the square of the scale factor.
- The student can explain the idea of "inner and out sums" and making finer grids to approximate the area of curved figures.
- The student can explain and use the method of linear approximations to curved figures to find perimeter. The student can find the area and circumference of circles given a radius, can describe what π is and give an approximation to its value, and can describe the method of approximating circles with inscribed and circumscribed regular polygons.
- The student can explain sine, cosine, and tangent in terms of invariant ratios in right triangles.
- The student can use ideas from triangle similarity to show the trigonometric ratios are invariant.
- Given two sidelengths of right triangles, the student can find the third length as well as the sine, cosine, and tangent of each angle.
- The student can express the height (and therefore the area) of triangles and parallelograms in terms of a side and an angle.
- The student sees the Pythagorean Theorem as a special case of a more general rule about the sides of triangles.

Projects

Problem 7 in Investigation 4.1 suggests a short project for students: creating a floor plan of a given space. It will require a lot of measurement of the space (and the funishings, windows, doors, etc.).

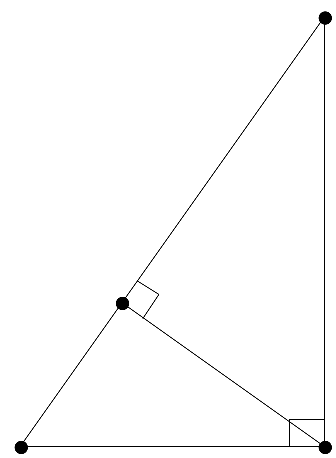
After Investigation 4.7: Is there some rectangle you can start with, for which you can do the ripping activity and end up with one set of similar rectangles? This is a good problem for integrating algebra. It amounts to finding x and y so that



$$\begin{aligned} \frac{x}{y} &= \frac{\frac{y}{2}}{x} \\ 2x^2 &= y^2 \\ x &= \frac{\sqrt{2}y}{2}. \end{aligned}$$

You can start with half a sheet of paper, so you can keep the other half as the "original" for comparison.

After Investigation 4.7: Here's another ripping activity. Start with any right triangle. Construct the altitude from the right angle to the hypotenuse and then tear, creating two new triangles. Are they scaled copies of the original? Explain. This requires at least the conjecture of AA similarity for triangles from Investigation 4.5. We are creating two triangles, each with a right angle and with one other angle shared with the original, so they are both similar to the original (and to each other).



A project of building a segment splitter would make an excellent assessment or extra credit assignment.

QUIZZES AND JOURNAL ENTRIES

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
4.2		Give students pairs of scaled figures and ask them to calculate the scale factor by measuring as in Problems 6 and 7.
4.5		Assess whether students can use mathematical tests to decide if two figures are scaled copies of each other. Use any of Problems 3, 6, 13, 19, and 20.

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
4.6	Present drawings that are close to, but not quite, scaled copies of each other. Ask students to decide why they are not scaled copies. What rule do they violate?	
4.10	For a major assessment, students must create scaled drawings and explain how they were done, either as a presentation or in writing.	
4.11	Students explain the Parallel and Side-Splitting Theorems and solve a problem like Problem 9.	
4.12	Ask for a detailed write-up of Problem 17 or a project investigation and write-up based on any of the *Take It Further* Problems. Students present proofs of the Parallel and Side-Splitting Theorems and answer related questions.	
4.13	Use short-proof write-ups (Problems 6 and 7) as assessment. Or use the section on Eratosthenes and circumference of the Earth for a bigger, project-type assessment.	
4.14		Choose one of Problems 9–11 as assessment.
4.15	Problem 11: You may want to give students a chance to turn in a rough draft and revise it. Ask students to present a proof of one of the similarity tests for triangles, from the students' own notes, with questions from you and the class.	Solve Problems 10 and 11, supplying drawings and explanation. Choose Problem 16 and one or two of Problems 6–10.
4.16	Use any of the constructions in the activity "A Constant-Area Rectangle."	Problems 19, 20, 35, or 54 Solve Problems 21 and 22 or Problems 24–26 by using geometry software.

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
4.17	Problems 19 and 20 For a challenging assessment, have students write a Logo program to calculate pi using one of the algorithms given in the Student Module (see Problem 15 for example). Or, have students write a program to investigate the probability of two numbers chosen at random being relatively prime (see the Perspective in the Student Module).	Problems 16–18
4.18	Problems 6, 8, 12, and 13: Choose two or three for a take-home assessment in which students carefully write up their work.	Problem 11
4.20	A well-designed write-up of the investigation and results of Investigation 4.20 could serve as a final assessment for this section.	Problems 8–14
4.21	Problem 7: Students should design their own version of a trigonometry table (without having seen any models). This is a good assessment for students who are comfortable with the ideas, but may come later for other classes.	
4.22	Choose two or three problems from Problems 10, 14, 15, and 17–19 and have students carefully write their answers.	If students did Problem 7, have them revise the table into sine, cosine, and tangent values.
4.23	Ask students to do a careful write-up of Investigation 4.22.	

Teaching Notes

Investigation **4.1**

Student Pages 1–4

INTRODUCTION TO MAPS AND BLUEPRINTS

OVERVIEW

Materials:
- rulers
- calculators

As an introduction to the concept of scaling, students use a map of downtown Seattle to compute distances between various locations. Students also work with the blueprints of a house to determine such measurements as the height of a chimney and the dimensions of an entire second floor.

Before starting the investigation, you might ask your students how they estimate distances when using a map. Many maps include a segment that represents some number of actual miles. In this case, people often put two of their fingertips at the endpoints of the segment and count how many segments fit between the two locations they care about. They then convert this number into miles by using the given scaling factor. Students should have a general familiarity with ratios and proportions.

TEACHING THE INVESTIGATION

"A Scale of One City" provides a good opportunity to review some of the basic properties of ratios and proportions. For instance, if 1 inch on the map represents 600 feet, then how many feet does 5 inches represent?

Maps are just one of several places where students might have encountered scaling. The "For Discussion" section asks them to think of other uses of scaling, such as when reading or creating blueprints.

In the "Working with Floor Plans" activity, students can use one of several strategies to solve Problem 4. They can use the edge of a sheet of paper to mark off a length that represents 3 feet (the distance from the top of the roof to the top of the chimney) and then count how many of these lengths fit into the desired measurement. Or, they can measure the actual length of the distance representing 3 feet and then set up a proportion to solve for the desired measurement. Both methods are worth mentioning if students do not suggest them.

The answers your students obtain in the blueprint activity will probably vary since their measurements may not be entirely precise.

ASSESSMENT AND HOMEWORK IDEAS.......

Maps

Students can use maps of their hometown to estimate such values as the distance between their house and their school. They can then check to see how their estimates compare to the actual distances.

Blueprints

While finding the blueprints for this investigation, we noticed that the lengths indicated on a drawing are not always consistent with each other. For instance, one length might be labeled “3 feet” while another length twice as long might be labeled “7 feet.” It might be interesting for your students to find other blueprints and check the consistency of the measurements.

For students who like to draw, Problem 7 makes a nice project.

Teaching Notes

Investigation **4.2**

Student Pages 5–11

WHAT IS A SCALE FACTOR?

OVERVIEW

Materials:
- rulers
- calculators

The investigation begins with students discussing the meaning of the term "scale factor." The remaining problems come in two varieties:

- Given a figure and a scale factor, compute the lengths of the figure after it has been scaled.
- Given two figures, compute the factor by which one was scaled to obtain the other.

The investigation concludes by examining how scaling affects the areas of squares and triangles and the volumes of cubes.

Students should have a basic understanding of fractions and decimals.

TEACHING THE INVESTIGATION

At the beginning of the investigation, students discuss what it means to scale a square by a factor of $\frac{1}{2}$. There is more than one way to interpret this statement: students might assume that scaling by $\frac{1}{2}$ means that the sidelengths of the square will become half as long, or they might think the area of the square will be halved. Both ideas are certainly valid, but the former meaning is the standard one.

Problem 2 reminds students that not all features of a object change when it is scaled (the square, for instance, maintains its 90° angles). The remaining "Checkpoint" problems give students practice in calculating with both fractional and decimal scaling factors.

Problem 6 gives pairs of scaled pictures and asks students to determine the scale factor needed to transform one into the other. Students will need rulers to measure distances and (if desired) calculators to convert the fractional answers into decimals. The problems provide a good opportunity to discuss how one calculates scale factor. In Part b, for example, there are numerous measurements one might take to determine the scaling factor of the 5-pointed stars. In each star, students might pick two of the tips (making sure to pick the same tips in both stars), measure the distance between both pairs of tips, and then calculate the ratios of the distances. Or, students might measure the length of a side of each star (making sure to pick corresponding sides) and then compute the ratios of the sidelengths. All methods should produce the same answer, with slight variations due to measuring inaccuracies.

The problems in the “Area and Volume” section provide a preview of a more detailed study of area and volume to come in Investigations 4.17–4.20.

ASSESSMENT AND HOMEWORK IDEAS.......

If you wish to provide students with more practice in basic scaling calculations, you can create problems similar to Problems 3 and 6.

Working with Directions: A Logo Activity

Overview

Materials:
- **Logo**
- **pencil and paper**

One way to think about changing the scale of a rectilinear figure is to multiply all the sidelengths by a fixed constant and to leave the angles alone. A good way to demonstrate this approach is to ask students to write a Logo procedure that draws a figure and then to adjust the figure so that it draws another copy, but with sides twice as long. For example, to change a procedure that draws a square of sidelength 100

```
to Square
  repeat 4 [fd 100 rt 90 ]
end
```

to one that draws a square of side length 200, you leave inputs to rt alone and change only the inputs to fd.

```
to BigSquare
  repeat 4 [fd 200 rt 90]
end
```

The idea is that the inputs to rt and lt control the *shape*, while the inputs to fd and bk control the *size*. One more level of abstraction makes the input to Square changeable:

```
to Square :s
  repeat 4 [fd :s rt 90]
end
```

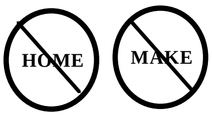
The house activity forces students to look at how the roles that inputs to fd play differ from those to rt. The cube activity gets at the idea that the ratio of the "diagonal" side in a cube to a "front" side is invariant under scaling.

This investigation guides students through the idea of scaling a figure by changing its size but not its shape in two contexts: drawing a house at different scales and drawing two-dimensional renditions of a cube.

For this investigation to work best, students should be familiar with writing procedures in Logo, including procedures that take inputs.

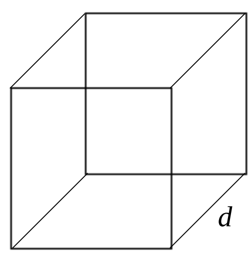
TEACHING THE INVESTIGATION

One of the authors used to put signs like this up in the computer lab:



Problems 3–6: Encourage students to write modular programs—one for the window, one for the roof, and so on. Try to see that students don't use particular screen coordinates, because they'll want to be able to draw houses of different sizes at various locations on the screen. In particular, don't let them include commands like "`home`" in their procedures.

Problems 7–11: The trick is to be able to express the length of the "diagonal" side in terms of the length of the "straight" side (this latter length will probably be the input to students' cube drawing procedure).



What's the ratio of d to s?

Let students figure this out themselves, either experimentally or by using the Pythagorean Theorem. They will discover that this ratio stays the same in different size cubes.

ASSESSMENT AND HOMEWORK IDEAS

As a follow-up to the Logo problems, you can ask students to write a series of procedures that will draw a scene and allow the user to "zoom in" on the scene by changing *one* input.

Teaching Notes

Investigation **4.4**

Student Pages 26–29

What Is a Well-Scaled Drawing?

Overview

In this investigation, we use "well-scaled" to describe copies that are drawn in proportion to the original. After this investigation, we will simply use "scaled" for the same concept.

Students compare a horse skeleton picture to four copies—some well-scaled and some not—to determine the features of a well-scaled drawing. They then are introduced to some of the numerical attributes of scaled copies (equal ratios of corresponding parts).

If you choose not to do Investigations 4.1–4.3, you can start the module with this investigation instead.

Teaching the Investigation

When students explain how they decided which horse skeletons were well-scaled copies of the original picture, you might hear answers like, "Picture 1 isn't a well-scaled copy because it's been stretched horizontally." Such answers are fine, but you might challenge your students a bit more by asking, "What if I'm not convinced the picture has been stretched? If I gave you a ruler and a protractor, what measurements could you take that would help convince me that picture 1 is not a well-scaled copy?" The purpose of asking this question is to get students thinking about the angle and ratio requirements necessary for two figures to be scaled copies. Students might, for instance, say that the angle of the horse's neck relative to the body is not the same in each picture.

Assessment and Homework Ideas

Problems 2 and 3 are good homework problems that focus on the numerical aspects of testing for scaled copies.

TESTING FOR SCALE

OVERVIEW

Materials:
- tracing paper
- rulers
- scissors
- calculators

In this investigation, students develop tests for checking whether pairs of rectangles, triangles, and polygons are scaled copies. Some tests involve measurement and calculation, while others rely strictly on visual checks.

Investigation 4.4 should be done before this one because it introduces students to the characteristics of scaled copies.

TEACHING THE INVESTIGATION

One of the best ways to begin learning about scaled polygons is to cut out, measure, and play with actual polygons. The investigation starts with simple polygons like rectangles and then asks students to develop scaling tests for arbitrary polygons. Here are some specific notes for the four sections of this investigation:

SCALED RECTANGLES

- Ask your students whether it is necessary to check angles to determine whether two rectangles are scaled copies.
- The rectangles in Problem 1 may look like scaled copies, but some measuring shows they are not.
- Problem 2 shows three pairs of rectangles and asks which pairs are scaled copies. You'll notice that there are some dashed lines included on these pictures with no explanation. We included these lines as a preview of things to come in Investigation 4.10. For now, if students ask you why the lines are there, ask them whether these lines can help to check if the rectangles are scaled copies. One possible conjecture is, "If the dashed lines that join the corresponding vertices all meet at a single point (as they do in b and c), then the rectangles are scaled copies."

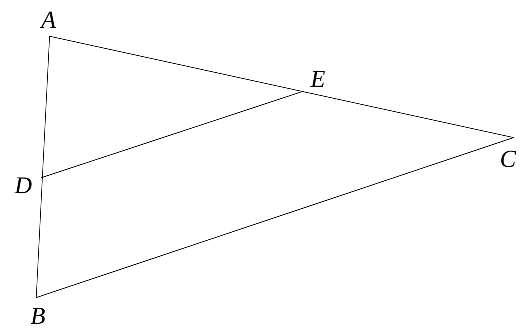
  It's possible, too, that if your students' measurements are a little off, they will say that *none* of the rectangle pairs are scaled copies. In this case, mention that, due to measuring inaccuracies, even well-scaled figures might not have the numbers work out perfectly.
- The "For Discussion" questions are important since they introduce the terms *corresponding sides* and *proportional*.

SCALED TRIANGLES

The two problems in this section give students a chance to explore what conditions must be checked to see if two triangles are scaled copies. Problem 6 points out that if you compute the inappropriate ratios, you might be misled into believing two triangles are not scaled copies.

CHECKING FOR SCALED TRIANGLES WITHOUT MEASURING

These problems encourage students to find ways to check for scaled triangles without taking any measurements. When students cut out and play with the scaled triangles in Problem 7, they might take the triangles and overlap them so that they meet at a common vertex A, as shown in this picture:



When the triangles are aligned this way, notice that:

- A pair of corresponding angles overlap perfectly at $\angle A$.
- $\overline{AD}$ and $\overline{AB}$, as well as $\overline{AE}$ and $\overline{AC}$ line up.
- $\overline{DE}$ is parallel to $\overline{BC}$.

These insights are used repeatedly throughout the rest of this module and will be discussed in Investigation 4.11.

SCALED POLYGONS

In this section, students develop tests to determine when two polygons are scaled copies.

ASSESSMENT AND HOMEWORK IDEAS.......

Draw other pairs of scaled and nonscaled rectangles, triangles, and polygons. Ask your students to determine whether they are scaled copies. Problems like 3, 6, 13, 19, and 20 make good assessment/homework questions.

Teaching Notes

Investigation **4.6**

Student Pages 42–43

The Many Faces of Scaling

Overview

This investigation reviews all of the criteria that students have developed for recognizing and checking for scaled copies. Students begin by looking at pairs of faces to pinpoint which features disqualify them from being scaled copies of each other. They then draw their own pictures to illustrate the same scaling tests.

This investigation reviews the scaling material from Investigations 4.4 and 4.5.

Teaching the Investigation

Knowing the characteristics of a scaled copy is an important skill that your students will use for the remainder of this module. Now is a good time to make sure that they understand this material.

Underneath each pair of faces, there is a description of a scaling requirement that is not followed. Ask your students to be specific and say what particular feature(s) of the faces do not conform to the given scaling requirement. Your students might notice that, for many pairs, there are several scaling criteria not met, aside from the one given. You can also ask your students to compare the list of scaling requirements given in this investigation to those developed in the previous two investigations.

Assessment and Homework Ideas

Problem 1, which asks students to make their own set of pictures, is a good homework question that gives students a chance to consolidate and review their understanding of what makes a scaled picture. For those students who enjoy drawing, this homework assignment might be turned into a small project.

Teaching Notes

Investigation
4.7

Student Pages 44–47

RECTANGLE DIAGONALS

OVERVIEW

Materials:
- **two $8\frac{1}{2}'' \times 11''$ rectangles for each student**
- **rulers**

In this investigation, students take a rectangular sheet of paper and repeatedly fold and tear it in half to form a collection of smaller rectangles. To test which rectangles are scaled copies without taking any measurements, students draw a diagonal line on each one, and then line up the rectangles so that they all meet at a common vertex. Through experimentation, students discover that those rectangles whose diagonals coincide are scaled copies. Students then extend this method to polygons to develop techniques that will be used extensively in the investigations to follow.

In Investigation 4.5, students measured and calculated the lengths and ratios of rectangle sides to determine if the rectangles were scaled copies. This investigation makes a good follow-up because it shows there are ways to check for scale that only require a visual check.

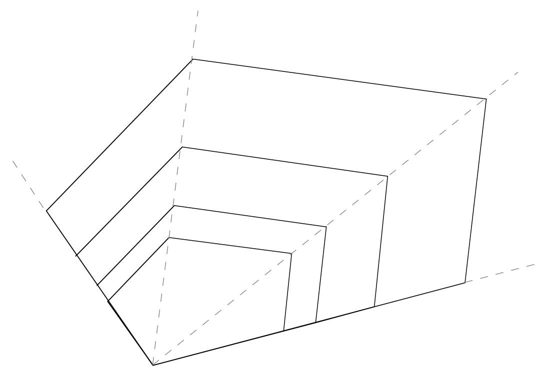
TEACHING THE INVESTIGATION

As students begin to fold and tear their rectangles, it's probably a good idea to have spare rectangles available in case they misinterpret the directions.

In order to make it simpler for students to see which of the torn pieces are scaled copies, you might consider the following adaptation: rather than start with a rectangle, use a square instead. By beginning with a square and repeatedly folding and tearing it in half, students will obtain a collection of scaled squares and a collection of scaled rectangles.

A very effective way to show that the "diagonals" of scaled polygons line up when they meet at a common vertex is to make a collection of overhead transparencies, each showing the same polygon at a different scale. Overlay the transparencies on top of each other so that the polygons meet at a common vertex. For the final transparency,

show the "diagonals" that pass through the corresponding vertices. The final collection of transparencies will look something like this:



Taking the time to do this polygon section is important, because it gives students an introduction to the dilation methods they will encounter later in the module.

ASSESSMENT AND HOMEWORK IDEAS.......

After students have developed the diagonal line test for rectangles, their homework assignment can be the "Write and Reflect" problem that asks them to explain *why* the folding and tearing technique creates two collections of scaled rectangles. The "Polygon Diagonals" section can also be given as homework since it is a direct extension of students' work with rectangles.

USING TECHNOLOGY ....

Rather than using overhead transparencies to show the diagonals of scaled polygons, you can achieve the same effect by drawing a polygon with geometry software. Pick a vertex as the center of dilation, and then repeatedly dilate the polygon from this vertex by different amounts. Finally, draw the diagonals to show that the vertices line up.

Teaching Notes

Investigation **4.8**

Student Pages 48–50

Light and Shadows: Projected Images

Overview

Materials:
- an overhead, slide, or movie projector
- a flashlight

Students study the images cast by movie and slide projectors, as well as the shadows cast by a flashlight, to gain insights into scaled images. The material serves as an introduction to the dilation methods to follow in upcoming investigations.

Teaching the Investigation

One of the key insights students make in this investigation is that the closer an object is to a light source, the larger its shadow will be. More specifically, if the distance from the light source to a wall is d and the distance from the light to the object is $\frac{1}{2}d$, then the shadow cast on the wall will be approximately twice as large as the object.

Give your class a chance to work with a projector in order to discover such information. An overhead projector will do, but students will probably get a better sense of the rules if they use a slide or movie projector. Have students demonstrate the general rules; then ask them to make some measurements to see if the rules can be made precise and numerical.

Assessment and Homework Ideas

The “Take It Further” question, which asks students to explore how copy machines make enlargements and reductions, makes a good mini-project.

CURVED OR STRAIGHT? JUST DILATE!

OVERVIEW

Materials:
- **geometry software**

Scaling figures that only have straight lines is fine, but how about those with curves? This investigation introduces a useful way to scale both polygons and curved figures—the *dilation* method.

Investigation 4.8 is a nice one to do before this one because its discussion of movie projectors and shadows motivates the dilation method introduced here.

TEACHING THE INVESTIGATION

It's important that your students get lots of practice with the dilation method because they will use it extensively in the following investigations. As your students work through this investigation, here are some points to keep in mind:

- Emphasize that, when the Student Module asks your students to "dilate a figure by a factor of r," it is asking them to scale the figure by r, using the dilation method. Later in the module, students will prove that dilation really does produce scaled copies.
- Remind students that they can pick the "center of dilation" point to be anywhere they would like. Its location only affects where the scaled image gets drawn. You can stress this idea by asking each student to pick a different center of dilation and then having students compare their work (see part b of Problem 1).
- One key insight to make is that a dilated copy is always in the same orientation as the original (see part b of Problem 2).
- Spend some time dilating figures only by a factor of 2 so that students get a feel for how the method works. Then, ask them to extend the method so that they can dilate by other factors (see Problem 3).
- If students draw too few rays when dilating a figure, they won't get a very detailed scaled copy. Encourage students to draw more rays if they're not happy with their results; at the same time, point out how this can be a potential limitation of scaling curved figures by hand.
- The mirror activity described at the end of this investigation gives surprising results—when students stand in front of a mirror and trace the image of their face onto it, the mirror picture is half the size of their face. The reason why this works (explained in the solutions) depends on dilation.

ASSESSMENT AND HOMEWORK IDEAS.......

Students should be able to explain the dilation method and use it to dilate points on any figure.

USING TECHNOLOGY ....

After students have practiced dilating with a pencil and ruler, they can move on to the section “Using Geometry Software to Dilate More Points.” This section explains a simple construction that simultaneously draws a figure along with its half-size copy (the results are similar to that of a pantograph). This is an excellent way for students to view a continuous, rather than discrete, version of the dilation method.

Teaching Notes

Investigation **4.10**

Student Pages 55–60

Ratio and Parallel Methods

Overview

Materials:

- large sheets of paper
- rulers
- geometry software (optional)

This investigation introduces two related methods for dilating a picture—the ratio and parallel methods. Students explore both techniques and then model the parallel method with geometry software.

Students should be familiar with the dilation material introduced in Investigation 4.9. It's possible to do this investigation by drawing on standard notebook-size paper, but the measuring and drawing will be easier if you provide slightly larger sheets.

Teaching the Investigation

Start by reviewing the dilation material in the previous investigation: How did students make scale drawings that were twice as large as the original picture? Half as large? The technique that students used is called the "ratio" method and is the first one described in this investigation.

In the Student Module, we give an example of how to scale a polygon *ABCDE* by $\frac{1}{2}$. Encourage students to draw their own polygons and then scale them by $\frac{1}{2}$ using the ratio method. Depending on where students place their center of dilation, they might end up with a scaled polygon that overlaps the original. One possibility is to give all of your students the same polygon to scale but allow each of them to choose their own center of dilation. If they then compare results, they will see that they all obtained scaled polygons, but each polygon is in a different location. As mentioned in Problem 2, these scaled polygons have sides that are parallel to the original polygon (an important fact that is used to introduce the "parallel" method).

Problem 7 is important because it shows what happens when we pick a center of dilation that coincides with a vertex of the polygon we wish to scale. Problem 8 is also worth covering because it illustrates a common mistake that students make when using dilation.

The second method for dilating polygons—the parallel method—requires students to be able to draw segments parallel to the sides of their original polygon. It's probably best to ask the students to just draw the segments as best they can, judging by eye what looks parallel. If you have geometry software available, you don't have to worry about this issue, because the software can automatically create parallel lines. Modeling

Ratio and Parallel Methods (continued)

the parallel method with geometry software also has another benefit: students can "slide" their dilated polygon back and forth and watch as it grows and shrinks, always remaining a scaled copy of the original polygon.

Assessment and Homework Ideas.......

Nearly any of these problems is suitable for homework and assessment purposes. We especially like the "Take It Further" problem; perhaps it can be used as an extra credit assignment.

Nested Triangles: Building Dilated Polygons

Overview

Materials:
- **geometry software**

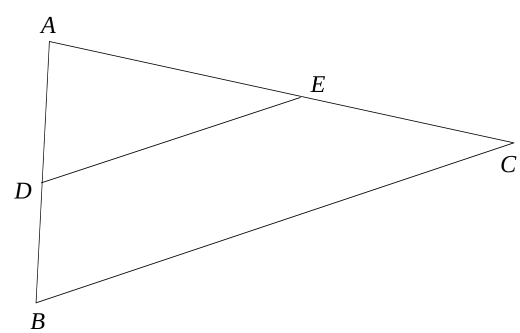
In this investigation, students perform experiments that lead to two important results about triangles: the Parallel and Side-Splitting Theorems.

Students should be familiar with the dilation methods for creating scaled drawings. This material was covered in Investigation 4.10.

Teaching the Investigation

The ratio and parallel methods both produce scale drawings, but why do they work? This investigation begins to answer that question by noting that each dilation picture is composed of *nested triangles*. A picture of nested triangles looks like this:

$\overline{DE}$ is parallel to $\overline{BC}$.



We provide students with three experiments to help them think about the properties of these special sets of triangles. All three experiments lead to the important Parallel and Side-Splitting Theorems. The verbal statements of these theorems can be rather difficult to understand, so we ask students in Problem 7 to make them more concrete by using a specific example. You should make sure that students feel comfortable working with these theorems, because they are central to many of the upcoming investigations. If your class is familiar with the Midline Theorem (which is discussed in the module *Habits of Mind* and proved in the module *The Cutting Edge*), students should discuss how it is related to the Parallel and Side-Splitting Theorems (see Problem 12).

ASSESSMENT AND HOMEWORK IDEAS.......

Problem 9 gives students practice in applying the results of the Parallel Theorem. For further practice, you can make up new problems just by changing the numbers.

WITHOUT TECHNOLOGY......................

Experiments Two and Three are designed to be used with geometry software. If you decide not to use software, you can give your students pictures of nested triangles (like the one on the previous page) and ask them to take measurements using a ruler.

Teaching Notes

Investigation **4.12**

Student Pages 69–86

SIDE-SPLITTING AND PARALLEL THEOREMS

OVERVIEW

Materials:
- **Geometry software is helpful, but not required.**

Students prove the Side-Splitting and Parallel Theorems by exploring a proof attributed to Euclid.

Next, students apply the Parallel Theorem to prove that the dilation methods introduced earlier in the module do indeed produce scaled copies of polygons.

Finally, students use the Side-Splitting Theorem to prove that the geometric figure formed by connecting the midpoints of an aribtrary quadrilateral is a parallelogram and to explore several related problems that build on the result.

Students should be familiar with the Side-Splitting and Parallel Theorems, as well as with the area formula for triangles.

Students can solve these problems without any knowledge of dilation, but those who did the dilation material in Investigations 4.9–4.11 will have the added bonus of seeing why it works.

TEACHING THE INVESTIGATION

In order for students to understand Euclid's proof, they need to be comfortable with the result stated in Problem 2: If two triangles have the same height, the ratio of their areas is the same as the ratio of their bases. The first three problems of the investigation give students practice in proving/applying this statement and reinforce the idea that any two triangles with the same length base and height have the same area.

We provide Euclid's proof of the Parallel Theorem essentially three times—the first time, we slowly give all of the details; the second time, we consolidate the steps to highlight its key aspects; and the third time, we give some indication of how one might discover the proof. You might work through the first proof with your class and then assign the other two proofs for at-home reading. Students who understand the proof well should be able to follow the same basic setup to prove the Side-Splitting Theorem (Problem 5).

The problems in the section "Using the Parallel and Side-Splitting Theorems" can, if necessary, stand on their own without any reference to earlier material. But for those classes who have worked their way through much of this module, these problems help

tie up some loose ends rather well. Specifically, Problem 13a proves the main result from Investigation 4.7.

Problem 13b relates back to Problem 2 in Investigation 4.7. In that problem, we asked students whether various pairs of rectangles were scaled copies of each other. We provided the dashed lines in the pictures, but didn't tell students why they were there. One conjecture that students might have made was, "If the dashed lines that join the corresponding vertices all meet at a single point, then the rectangles are scaled copies." Now students can prove that this is indeed true (if they haven't done the earlier investigation, you can show them the problem from it).

Problem 14 proves why "nested triangles" are scaled copies of each other and Problem 15 shows why the parallel method for dilation works (actually, *every* problem in this investigation helps to prove why the parallel method for dilation works—each illustration shows a figure that has been dilated).

If students have seen this result before in the *Habits of Mind* module, the focus here should be on proof using the Side-Splitting Theorem. If not, you may want to allow more time for investigation and conjecture before tackling the proof.

Although geometry software is not necessary for the section "Midpoints in Quadrilaterals", it fits nicely with many of its problems. In particular, it makes the result from Problem 16 quite striking. If students draw an arbitrary quadrilateral with their software and then connect its midpoints, they will see that the inner quadrilateral is a parallelogram regardless of how they drag the vertices of the outer quadrilateral. Problem 17 asks them to prove this result. You might give your students a hint by suggesting they draw the segment connecting points A and C in their outer quadrilateral. Notice that the illustration accompanying this problem shows three different configurations of quadrilateral $ABCD$, two of which look a bit out of the ordinary. It's interesting to ask students whether they consider all three figures to be quadrilaterals and if they can prove the midpoint result for all of them.

The remainder of the problems in this section are all variations on the same theme of connecting the midpoints of quadrilaterals. If time permits, do at least a few of them, since they provide students with a common thread of mathematical reasoning running through a series of problems.

ASSESSMENT AND HOMEWORK IDEAS.......

We don't often have a chance to show our students how the same mathematical theme can be used to solve a variety of interconnected questions. The first "Take It Further" section consists of a very nice collection of challenging problems (with hints) that

revolve around the theme developed early in the investigation: triangles with the same base and height have the same area.

Problems 15, 17, and 18 are good for homework and/or tests, while the remainder of the problems are more appropriate for projects.

For more extensions of midpoints in quadrilaterals, check out the article, “The Sidesplitting Story of a Midpoint Polygon” by Y. David Gau and Lindsay A. Tartre *The Mathematics Teacher*, 87 (April, 1994), 249–256.

Teaching Notes

Investigation
4.13

Student Pages 87–91

Historical Perspective: Parallel Lines

Overview

Deduction, proof, and axiom systems play a central role in mathematics. This investigation tries to convey some of the uses of proof in mathematics, some of the historical reasons that the deductive tradition took hold, and how axiomatics can bring coherence and structure to mathematical results. The context for this discussion is Euclid's *The Elements*. The investigation concludes with an application of the results to a practical question solved quite neatly by the Greek mathematician Eratosthenes: How can one measure the circumference of the Earth?

Teaching the Investigation

Starting with Euclid's five postulates, the investigation develops theorems about vertical angles, exterior angles, and alternate interior angles. This collection of axioms and theorems is one of the most famous in all of mathematics, and it provides a perfect example of a self-contained logical system. Chances are, you will need to provide your class with some assistance in doing these problems, as they use such methods as *proof by contradiction*.

Even if you decide not to spend time with this material, you should try to cover the application problem at the end of the investigation—finding the circumference of the Earth. Eratosthenes's method is very clever and gives a remarkably good estimate for the circumference.

Assessment and Homework Ideas

Consider assigning the section on Eratosthenes as homework or a short project. The Student Module gives a reference to the book *Poetry of the Universe*, where students can read more about Eratosthenes's method and other mathematics relating to the Earth and astronomy.

Defining Similarity

Overview

Students are introduced to two definitions of *similarity* and learn about the similarity notation "∼".

One of the similarity definitions given in this investigation relies on students being familiar with the term *dilation*. If you have not done any of the earlier dilation investigations, you can concentrate on the other similarity definition.

Teaching the Investigation

The first part of this investigation gives two possible definitions of similarity—one involving scaling and the other dilation. The dilation definition isn't quite complete. It says that two figures are similar if one is a dilation of the other. While this is true, it limits similar figures to those pictures that are both oriented in the same way (recall that dilation does not change the orientation of a figure). To help students see that this definition does not encompass all similar figures, Problems 3 through 5 show a horse and a scaled copy of it that are pointing in opposite directions. Students are asked if they can modify the definition to include this case.

In Problem 12, students are asked to prove that two triangles—one with sides twice as long as the other—are similar. We provide one proof of their similarity in the "Ways to Think About It" section immediately following the problem. This is an important proof to understand, as it is the basis of the SSS triangle similarity theorem introduced in the next investigation.

Assessment and Homework Ideas

One of our favorite problems to give people (not just students!) is the napkin challenge in Problem 13. For more traditional assessment, assign Problems 9 and 10.

Similar Triangles

Overview

Students prove the AA, SAS, and SSS triangle similarity theorems and apply them to a variety of problems.

In order to understand the proofs, students should be familiar with the following definitions of similar triangles:

- Two triangles are similar if their corresponding angles are congruent and their corresponding sides are proportional.
- Two triangles are similar if one is congruent to a dilation of the other.

The first definition appears in the solution for Problem 9 of Investigation 4.5 on page 18 of the Solution Resource; the second definition is discussed in Investigation 4.14.

Teaching the Investigation

The proofs of the AA, SAS, and SSS theorems often receive very minimal attention in geometry texts. Since the theorems form the basis for so many problems that students will solve in the future (including trigonometry), you should try to find time to cover them.

After students have developed their own triangle similarity tests (Problem 1), you can start with the AA theorem. You'll notice that in the Student Module we refer to this theorem as "AAA" rather than "AA." That's because we want students to figure out for themselves that it's not necessary to write "AAA," since just two angles will automatically determine the measure of the third angle (see Problem 2).

If you carefully review the provided written proof of the AA theorem with your class, your students can then do the SAS proof themselves as the proofs are nearly identical. The proof of the SSS theorem is slightly more complicated, and uses the dilation definition of similarity.

Assessment and Homework Ideas

The application problems that follow each proof are all well-suited for either homework or assessment. In particular, we like Problem 20 because students have to figure out for themselves which theorem to apply.

Teaching Notes

Investigation **4.16**

Student Pages 110–140

USING SIMILARITY

This investigation contains five activities. Because of the independent nature of these activities, we are providing separate Teaching Notes for each activity.

Calculating Distances and Heights

OVERVIEW

Materials:
- calculators
- tape measures

Students explore three different "real-world" applications of similarity.

Students should be familiar with the AA Triangle Similarity Theorem from Investigation 4.15. For the application "Tiny Planets," familiarity with scientific notation and facility in calculating with numbers written in scientific notation (by hand or with a calculator) are required.

TEACHING THE INVESTIGATION

We've included three different applications of similarity in this investigation that allow one to find unknown distances and heights:

A SEA STORY

By comparing the length of an outstretched thumb to the height of an object it covers (in this case, a mountain), students are able to compute the distance to the mountain. Problem 3 encourages students to try this technique with an actual object to see how well it works.

A "SHADY" METHOD

This is the classic application of similarity, using the length of a shadow to figure out the height of a tree. Students can try this method for themselves, perhaps finding the height of their school flagpole (yet another classic application of this theory).

TINY PLANETS

This very timely question asks students to figure out how big a planet would appear to the naked eye if we found a planet of our nearest star.

USING SIMILARITY (continued)

For all of these applications, students will find that their calculations are only approximate. You might discuss with them the factors that contribute to the inaccuracies.

ASSESSMENT AND HOMEWORK IDEAS. . . . . . .

Each of these three applications would make a good project for students who are interested in practical application of similarity.

Segment Splitters

OVERVIEW

Materials:
- **blackline master for Experiment One**
- **straightedges**
- **lined notebook paper**
- **blank overhead transparencies**
- **erasable overhead markers**

This investigation provides several experiments that allow students to split a segment into any number of congruent pieces without taking a single measurement. The investigation also includes an historical account of a scientist who designed and patented her own segment-splitting device.

In order to prove why these segment-splitting techniques work, students will need to be familiar with the Parallel Theorem from Investigations 4.11 and 4.12.

TEACHING THE INVESTIGATION

The two experiments described in this investigation are really surprising: Imagine being able to divide a segment into any number of congruent pieces without taking a single measurement! To give your students the excitement of this discovery, it might be a good idea to jump right into these experiments *without* giving an introduction like, "Today you will be dividing a segment into congruent pieces by . . . "

These experiments work well when done by pairs of students, with half the class working on Experiment One and the other half doing Experiment Two. Each group can then report its findings to the rest of the class.

A blackline master for Experiment One is included at the end of the notes for this activity.

ASSESSMENT AND HOMEWORK IDEAS

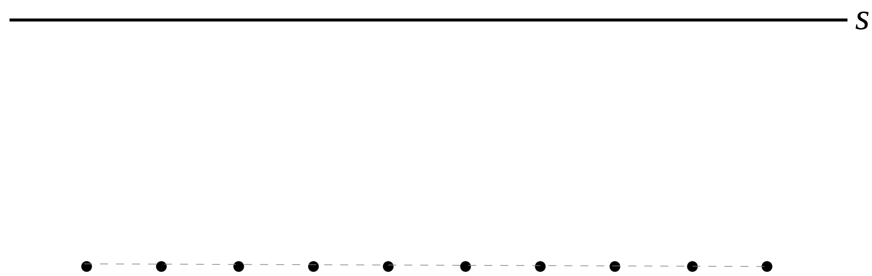
Problem 19 in the "Why Do the Methods Work?" section can be given as homework after students have spent the class period trying both experiments. You might also encourage your students to go home and explain the experiments to their parents or siblings, who will probably be equally surprised by the results.

The Perspective essay on Sarah Marks is a nice assignment to give for at-home reading. It asks students to figure out how Sarah Marks's segment splitter is used, as well as why it works. Some students might also be interested in constructing their own model of the segment splitter as a project, and then demonstrating it to the rest of the class.

ADDITIONAL RESOURCES

Just as we were putting the finishing touches on this module, there came news that two students from Greens Farms Academy in Connecticut devised their own method for splitting a segment into any number of congruent parts using geometry software. For a complete account of their method, see the article "Euclid, Fibonacci, Sketchpad" by Daniel C. Litchfield and David A. Goldenheim, *The Mathematics Teacher* (January, 1997), 8–12.

USING SIMILARITY (continued)



A Constant-Area Rectangle

OVERVIEW

Materials:
- geometry software

In this activity, students prove a surprising result about chords of a circle that all pass through a common point—the *power-of-a-point* theorem. Students then apply the theorem to create what we call a *constant-area rectangle* with geometry software. A constant-area rectangle is one whose dimensions can change but whose area remains fixed as a vertex is dragged.

In order to prove the power-of-a-point theorem, students will need to know that inscribed angles in a circle that intercept the same arc are equal in measure (material not covered in this module). They also will need to be familiar with the AA theorem for triangle similarity.

Typically, the power-of-a-point theorem appears in geometry books as a mere curiosity. Students learn the theorem, prove it, use it in some numerical problems, and then move on. It probably leaves no lasting impression. This activity attempts to remedy that situation by showing how the theorem can be used to create an interesting animated object with geometry software—a constant-area rectangle.

Specifically, the geometry that lies behind the power-of-a-point theorem becomes the "engine" that drives the movement of the constant-area rectangle. By setting the theorem in motion, students are able to uncover relationships that its static counterpart in a textbook cannot reveal.

TEACHING THE INVESTIGATION

The first, and probably most important, thing to say about this investigation is that you should definitely work it through yourself before trying it with your students. We've adopted a novel approach to teaching the power-of-a-point theorem (as described above) that we think you'll enjoy, but it does require some preparation on your part.

The investigation begins with a brief review of the distinction between drawing a rectangle with geometry software and constructing one. It assumes a fair degree of familiarity with the construction tools of geometry software, so students should be comfortable with the idea of making geometric constructions with the software. If your

your class is not at this level or you do not have sufficient computer lab time, you can show them pre-made sketches of constant-area rectangles instead. You'll find a downloadable sketch created with *The Geometer's Sketchpad*® at the *Connected Geometry* web site: http://www.edc.org/LTT/ConnGeo. You will also find Java animations of the sketch at this site.

Proving the power-of-a-point theorem (Problem 35) requires some knowledge of inscribed angles in circles. We chose not to include background information about inscribed angles in the lesson because each class will have different needs. Depending on the background of your students, it might be just a matter of a quick review of inscribed angles. For other classes, you might need to devote more time to the topic.

In order to provide a transition from the power-of-a-point theorem to constant-area rectangles constructed with geometry software, we included Scenario One under the "Making the Connection" section. This scenario asks students to think about how they can use their power-of-a-point results to build wooden rectangles all sharing the same area. By working through this scenario, students will have a smooth introduction to the task of building a constant-area rectangle with geometry software.

ASSESSMENT AND HOMEWORK IDEAS.......

A good way to assess your students' understanding of this investigation is to ask them to explain to you (or a classmate) how and why their geometry software construction works (Problem 40). For students who are very interested in geometry software, Problem 44, which asks them to construct a constant-perimeter rectangle, can make an excellent project.

WITHOUT TECHNOLOGY ....

If you choose to do this investigation without geometry software, then you can complete everything up to and including Scenario One.

ADDITIONAL RESOURCES

For more background information on this investigation, see the article “Theorems in Motion: Using Dynamic Geometry to Gain Fresh Insights” by David P. Scher, *The Mathematics Teacher* 89 (April, 1996), 330–332.

The Geometric Mean

OVERVIEW

Materials:
- geometry software

This investigation is a continuation of “A Constant-Area Rectangle” and uses the power-of-a-point construction to introduce the concept of *geometric mean*. Given two segments of lengths a and b, students construct their geometric mean—a segment of length $\sqrt{ab}$.

You will need to do the activity “A Constant-Area Rectangle” before moving on to this one.

TEACHING THE INVESTIGATION

The investigation begins with several rectangle challenges. Students are asked to construct the missing side of a rectangle so that the completed rectangle has an area equal to that of a given rectangle. The “Ways to Think About It” section shows how this task can be done; as you’ll see, the method involves applying the results from the power-of-a-point material.

All of this builds up to Problem 47: using the power of a point to draw a square with the same area as an $a \times b$ rectangle. As explained in the Student Module, the side of this square is called the *geometric mean* of a and b.

Many of you are probably familiar with the method shown in Problem 54 for introducing the geometric mean. It would be very instructive to cover this method with your students and contrast it to the power-of-a-point method they’ve just done.

The investigation concludes with a repeat of the challenge from “A Constant-Area Rectangle”: How can you construct a constant-area rectangle using geometry software? This time, however, students are asked to build the rectangle using the geometric mean results developed in this investigation. You’ll find a downloadable sketch of this construction created with *The Geometer’s Sketchpad*, as well as a Java animation, at the *Connected Geometry* web site: http://www.edc.org/LTT/ConnGeo.

ASSESSMENT AND HOMEWORK IDEAS.......

Problems 52 and 53 are both good ways to assess students' understanding of the geometric mean. Problem 57, which asks students to build a constant-area rectangle, is an excellent project for those students with some knowledge of geometry software.

ADDITIONAL RESOURCES ....

In Problem 59, a collection of similar triangles is used to create lengths that form a geometric sequence. This construction dates back to Descartes and can be extended to create logarithmic curves with geometry software. For a wonderful discussion of the construction and its potential uses in the classroom see the article "Drawing Logarithmic Curves with *Geometer's Sketchpad*: A Method Inspired by Historical Sources" by David Dennis, *Geometry Turned On* (MAA Notes #41, 1997), pp. 147–156.

No Measuring, Please!

OVERVIEW

Materials:
- a straightedge
- a compass
- lined notebook paper

In this investigation, students are given a picture of two segments along with a unit segment. They are then challenged to construct other segments (such as their product, sum, and quotient) in a purely geometric way, without taking any measurements.

In order to solve these problems, students should be comfortable using the Parallel Theorem and the Pythagorean Theorem. Some of the problems use construction techniques developed in earlier activities of this investigation. See the Solution Resource for details.

TEACHING THE INVESTIGATION

This is a very elegant activity because all of the results can be obtained without taking any measurements. You probably won't be able to do all of the problems at once since they rely on techniques that extend over several previous investigations. You might return to this investigation at several points during the study of similarity and do those problems that are appropriate for the material you've covered.

The techniques needed to solve problems like 64 and 65 might not occur to your students without some hints. It might be a good idea to demonstrate the solution for one of these problems and ask your students to do the other.

Teaching Notes

Investigation **4.17**

Student Pages 141–146

AREAS OF SIMILAR POLYGONS

OVERVIEW

Materials:
- rulers
- scissors

Investigations 4.17–4.20 serve several functions:

- to explore how the areas of polygons and closed curves are affected by scaling;
- to develop methods for computing the areas and perimeters of closed curves;
- to introduce π and derive the area and circumference formulas for circles.

The material in these investigations is tightly connected, and many of the results are built from earlier results.

The main concepts developed in this investigation are:

- When a rectangle or triangle is scaled by a factor of r, the ratio of the area of the scaled copy to the original is r^2.
- When an arbitrary polygon is scaled by r, the ratio of the area of the scaled copy to the original is r^2. This is shown by dividing the polygon into triangles.
- The formula "area $= \frac{1}{2}$(perimeter)(apothem)" can be used to find the area of a regular polygon.

In this investigation, students begin by discovering how the areas of rectangles and triangles are affected by scaling. They then use these results to address the same issue for general polygons. The apothem of a regular polygon is introduced, along with a way to use it to calculate the areas of regular polygons.

Students should be able to scale rectangles and triangles and have some knowledge of elementary algebra.

TEACHING THE INVESTIGATION

One of the most effective ways for students to see how a change in scale affects the area of a rectangle or triangle is to actually draw the figure, scale it, and see how many copies of the scaled figure fit inside the original. Students either draw and scale the figures by hand or use geometry software to do the scaling for them.

Problem 8 will probably require some explanation from you. The area of Polygon 1 is $a + b + c + d$, while the area of Polygon 2 is $ar^2 + br^2 + cr^2 + dr^2$. In order for students to see that the ratio of the two areas is r^2, you might need to point out that the area of Polygon 2 can be rewritten as $r^2(a + b + c + d)$.

AREAS OF SIMILAR POLYGONS (continued)

The apothem and the area formula for regular polygons are introduced here because they will be used in Investigation 4.18 to derive a relationship between the area and circumference of a circle. If you're short on time, you can skip the apothem and return to it later.

ASSESSMENT AND HOMEWORK IDEAS.......

Problem 10 is a good example of how the area/scaling concept can be applied to an actual situation. Problem 11 is also valuable because students get the chance to measure a polygon, calculate its area, and then figure out what happens to the area when the polygon is scaled.

The "Take It Further" section gives students another practical application of area and scaling—constructing scale drawings of two plots of land to determine which has the larger area. Making the scale drawings and calculating their areas involves a fair amount of work and can make an excellent project. Students can use either a pencil, ruler, and protractor or geometry software to create the scale drawings.

Teaching Notes

Investigation **4.18**

Student Pages 147–154

Areas of Blobs and Circles

Overview

Materials:
- **graph paper**
- **a large piece of paper for Problem 7**

The main concepts developed in this investigation are:

- How can we find the areas of figures that have curves? We can obtain better and better approximations for the area of a curved figure by laying finer and finer ruled graph paper on top of it and counting how many squares cover the figure.
- In particular, graph paper can be used to estimate the area of a unit circle.
- When a curved figure is scaled by r, the ratio of the scaled copy to the original is r^2.

In Investigation 4.17, students examined how the areas of polygons change when they are scaled. This investigation extends the process to closed curves and circles, and asks how their area changes when they are scaled. Fundamental concepts from calculus, such as sequences, approximations, and limits, are introduced in an intuitive way to answer the questions.

Investigation 4.17 should precede this one.

Teaching the Investigation

This investigation serves as groundwork for many of the ideas that your students will eventually encounter if they study calculus. It's intended to be an informal, intuitive introduction to concepts like sequences and limits; no previous knowledge of these ideas is assumed.

Before introducing the grid method described in the Student Module, be sure to ask students to come up with their own ideas for how to estimate the area of the "blob." Once you begin the grid method, your students should be able to suggest ways to make it give better and better approximations. When students reach calculus, they do not always get the chance to actually take measurements and try out this grid method for approximating area. By giving them the chance to do so here, you'll help them build their intuition so they'll be ready for a more formal approach in calculus.

Rather than just tell students that the area formula for a circle is πr^2, the Student Module develops it from scratch. If students have derived this formula in Investigation 3.4 of the module *The Cutting Edge*, you may want to point out the similarities between the two derivations. Problem 7 in this investigation begins the development of the formula. It asks students to draw a circle with a one-foot radius and estimate its area.

Once your students know the area of a circle with radius one, they will be able to use this value to figure out the area of *any* circle in Investigation 4.20.

Problem 8 is an informal proof that when a closed curve is scaled by r, its area changes by r^2. The basic idea behind this proof is that a curve can be approximated by a grid of squares and each individual square grows by a factor of r^2 when the curve itself is scaled.

ASSESSMENT AND HOMEWORK IDEAS.......

For homework, you can give your students another "blob" and ask them to approximate its area as best they can using the grid method. On a test, students should be able to explain and use the grid method, and describe how to obtain better and better area estimates.

Teaching Notes

Investigation **4.19**

Student Pages 155–162

PERIMETERS OF BLOBS AND CIRCLES

OVERVIEW

Materials:
- rulers

The main concepts developed in this investigation are:

- To find the perimeter of a closed curve (affectionately called a blob in the Student Module), you can approximate it by a series of straight lines.
- To find the circumference of a circle, you can inscribe and circumscribe polygons with more and more sides in and about the circle. The perimeters of these polygons serve as upper and lower bounds for the circumference and approach a common value as the number of polygon sides grows.
- The formula "area of circle = $\frac{1}{2}$ (circumference)(radius)" is developed as an extension of the polygon area formula from Investigation 4.17.

This investigation introduces techniques for finding the perimeters of curves and approximating the circumferences of circles. It also develops a formula that relates the area of a circle to its circumference.

In order to do the section "Connecting Area and Circumference," students will need to have studied the polygon area formula in Investigation 4.17.

TEACHING THE INVESTIGATION

This investigation is an informal introduction to perimeter concepts that your students will eventually encounter if they study calculus. The basic question here is: How can you estimate the perimeter of a curve and then improve your estimate? The approach—estimating the perimeter with a series of straight lines—is one that your students should be able to suggest. Be sure to give your students the chance to try this method out with either the blob provided in the Student Module or another curve that you provide (perhaps a larger one). When students reach calculus, they do not always get the chance to actually take measurements and try out this approximation technique. By giving them the chance to do so here, you will help them build their intuition so they will be ready for a more formal approach in calculus.

A similar, but not identical, approach to perimeter and area for circles is given in the module *The Cutting Edge*.

The section "Perimeters of Circles" extends this perimeter estimation technique to circles and introduces the important idea of circumscribing and inscribing polygons in and about a circle. Again, being able to actually try out this idea is the most important benefit students will obtain from this section. For convenience sake, pictures of 4-sided, 8-sided, and 16-sided polygons inscribed in and circumscribed about a circle

are provided so that students only have to take measurements. They can also use the polygon scripts provided with geometry software to draw the polygons.

Perhaps the most challenging part of this investigation is the "Connecting Area and Circumference" section. Here's a capsule summary of its main ideas:

- The area formula for a regular polygon is $A = \frac{1}{2}Pa$ (where P is the perimeter and a is the apothem).
- Since we can estimate the area of a circle by inscribing a polygon in it, we should be able to use the polygon area formula to estimate the area of a circle. This approximation will get better and better as the number of polygon sides increases.
- When the inscribed polygon has many sides, its perimeter is essentially the circumference of the circle, and its apothem approaches the circle's radius. Thus, we can replace P by C (circumference) and a by r (radius) to obtain the circle area formula $A = \frac{1}{2}Cr$.

ASSESSMENT AND HOMEWORK IDEAS.......

For homework or a test, you can give your students another "blob" and ask them to approximate its perimeter. They should be able to explain their method and give ways to improve it. Your students should also be able to explain the method used for approximating the circumference of a circle.

Teaching Notes

Investigation 4.20

Student Pages 163–172

Circles and 3.14159265358979323...

Overview

The main concepts developed in this investigation are:

- If a circle is scaled by r, then the ratio of the area of the scaled copy to the original is r^2. This gives a general formula for the area of a circle: If the area of a circle with radius 1 is K, then the area of a circle with radius r is Kr^2.
- The value of K is approximately 3.14 and is known as π.
- Now that we know the formula for the area of a circle, we can substitute it into "area of circle $= \frac{1}{2}$(circumference)(radius)" and find a general formula for the circumference of a circle.

This investigation derives the area and circumference formulas for circles and gives a historical introduction to π. It is a capstone to the section covering areas, circles, and π. It uses information from Investigations 4.17–4.20.

Teaching the Investigation

In order to derive the area formula for a circle, students first need to know the area of a circle with radius one foot (Problem 7 from Investigation 4.19). Now is a good time to do (or review) this problem if your students have not done it already.

This investigation builds up to the area formula of a circle in increments. First, Problem 3 uses the area of a circle with radius 1 foot to find the area of *any* circle. This is an important problem, and you'll probably want to go through it carefully. Some of its key points are:

- The area of a circle with radius one foot is approximately 3.1 square feet.
- When a curved figure (in particular, a circle) is scaled by a factor of r, its area changes by r^2.
- Thus, a circle with a radius of 5 feet, for example, has an area of approximately $3.1(5)^2$ square feet.

Students are then asked to prove Theorem 4.10, which says that the area of a circle with radius r is Kr^2, where K is the area of a circle with radius 1. Notice that we don't immediately call the area of a unit circle "π." We prefer to stress the idea that π is a constant, and, aside from its fancy appearance, there is nothing particularly magical about it.

ASSESSMENT AND HOMEWORK IDEAS.......

Asking students to explain how they derived the formula for the area of a circle is a nice idea for a homework or test question. Problem 11, which asks students to predict and then calculate some measurements on a tennis ball canister, is a good question with a surprising result.

The historical overview on π can lead to some student projects. A sidenote in the text makes reference to an excellent article on π in *The New Yorker*. It's a substantial, humorous article that students can read and then report on to the class. The book *A History of Pi* (also mentioned in a sidenote) is another good source for a project report.

You can also direct your students to the World Wide Web page

http://www.go2net.com/internet/useless/useless/pi.html,

which contains some fun (and not entirely useless) information about pi.

Another book about π is *The Joy of Pi* by David Blatner, published by Walker and Company. A related Web site,

http://www.joyofpi.com,

contains information about pi and links to many other pi-related sites on the World Wide Web.

So Many Triangles, So Little Time

Overview

Materials:
- calculators

As an introduction to the methods of trigonometry, students explore three different ways to calculate the lengths of roofs.

Students should know that the ratios of corresponding sides of similar triangles are equal.

Teaching the Investigation

Sometimes, when students are introduced to trigonometry, they forget that it is really nothing more than a way to standardize and give names to the constant ratios found in similar triangles. With this in mind, the investigation sets out to do the following:

- It motivates the introduction of trigonometry by showing that, if there were a standardized way to solve for unknown sides of right triangles, it could speed up the entire process.
- It offers three "proposals" for standardizing the solution process, one of which (Proposal 3) is the usual trigonometric method of looking at ratios.
- By giving more than one proposal, it shows that there is nothing particularly magical about the trigonometric method. In fact, the investigation asks students to decide for *themselves* which method they like best.

By delaying the discussion of trigonometry to the next investigation, students will already feel comfortable with ratios and regard the names cosine, sine, and tangent as just handy labels for what they already know.

You might try dividing your class into three groups and making each group responsible for trying one of the three proposals. They can then report their results to the class, and discuss which method they find easiest.

Assessment and Homework Ideas

Problem 7 is a good way to bring the investigation to a close since it asks students to design their own reference chart for solving right triangle problems. You can also give students more practice with each of the three different methods by changing the numbers in the problems.

TRIGONOMETRY

OVERVIEW

Materials:
- calculators

Students are introduced to the trigonometric words “sine,” “cosine,” and “tangent” and then solve right triangle problems using this new terminology.

In order to understand the ideas behind trigonometry, students should be familiar with the basic properties of similar triangles (such as equal ratios of corresponding sides) covered earlier in this module. While not entirely necessary, it would be helpful if students have also done Investigation 4.13.

TEACHING THE INVESTIGATION

Problems 1–3 look at two right triangles, both of which have one acute angle of 27°. These problems stress that even though the overall sizes of the triangles are different, the ratios of their sides are the same. Assuming that students are comfortable with this idea, you can then move on to introduce the trigonometric names given to these ratios. You should make sure that students are able to find trigonometric values using their calculators; be on the lookout for students whose calculators are set in radian rather than degree mode.

ASSESSMENT AND HOMEWORK IDEAS

Any of the problems are appropriate for assessment or homework, depending on which aspects of trigonometry you wish to emphasize. Problems 4 through 10 give students practice applying the trigonometric ratios and terminology to basic right triangle questions. Problems 11 and 12 give real-world applications of trigonometry. Problem 14 introduces students to the “special” angles (30°, 45°, and 60°) for which they can find the exact values of sine, cosine, and tangent.

Teaching Notes

Investigation **4.23**

Student Pages 186–187

An Area Formula for Triangles

Overview

Materials:

- calculators

Students use trigonometry to derive the triangle area formula $A = \frac{1}{2}ab\sin\theta$.

Students should know the standard area formula for triangles, Area = $\frac{1}{2}$(base)(height), and be familiar with the trigonometric ratios introduced in Investigation 4.22.

Teaching the Investigation

Problem 1 asks students how to find the area of a triangle when they are given the lengths of two sides and the measure of the included angle. The problem gives specific numbers so that students do not have to generalize the procedure all at once. Before moving on to Problem 2, which asks students to find a general formula, you might give your students a few more problems with specific numbers so they start to see the pattern.

Assessment and Homework Ideas

It is important to do Problem 3, which asks students to extend the triangle area formula to parallelograms, since the results will be used in the next investigation.

Teaching Notes

Investigation **4.24**

Student Pages 188–195

EXTENDING THE PYTHAGOREAN THEOREM

OVERVIEW

Materials:
- **geometry software**

Students' typical introduction to the The Law of Cosines is a highly algebraic one: After lots of symbol manipulating, they arrive at the formula $c^2 = a^2 + b^2 - 2ab\cos C$. There's something not entirely satisfying about this method. The Law of Cosines formula is very close to the Pythagorean Theorem ($a^2 + b^2 = c^2$), but the Pythagorean Theorem has very nice geometric dissection arguments to prove it. (See the *Connected Geometry* module *The Cutting Edge*.) Given the similarity of the Pythagorean Theorem and the Law of Cosines, the authors wondered whether there was also a geometric dissection to prove the Law of Cosines. We found one!

The *Connected Geometry* module *Habits of Mind* contains another intriguing derivation of the Law of Cosines in the *Teaching Notes* for Investigation 1.15.

For this investigation to work best, students should be familiar with the Pythagorean Theorem. (See the *Connected Geometry* module *The Cutting Edge* for a full discussion of the theorem.)

TEACHING THE INVESTIGATION

Begin with Problems 1 and 2, which ask students to experiment with nonright triangles to see if the Pythagorean relationship still holds. These problems are followed by "A Recipe for Extending the Pythagorean Theorem." This "recipe" is a geometry software construction that requires students to be able to construct squares and parallelograms. Encourage your students to keep referring to the illustration that accompanies this construction so that they don't construct a square or parallelogram facing the wrong direction. If students find the construction too difficult, you might provide them with it already made.

By experimenting with the construction (Problem 4), students should be able to conjecture that the area of square *ADEB* is equal to the sum of the areas of the two smaller squares and two parallelograms. This is the main conjecture that students will develop in this investigation.

Now that students know the relationship between these areas, all that remains is to algebraically determine their values in Problems 7 and 8. Finding the areas of the squares is straightforward, but calculating the parallelograms' areas requires using the result from Problem 3 in Investigation 4.23. Theorem 4.13 brings this information all together in one formula.

Problem 11 asks students to apply the theorem to two triangles, finding the lengths of unknown sides. "Take It Further" Problem 13 provides a nice tie-in to the Pythagorean

Theorem. By making $\triangle ABC$ a right triangle in the software construction, the two parallelograms disappear, and you're left with a picture that nicely illustrates a dissection proof of the Pythagorean Theorem. (See the Solutions Resource for details.)

You'll notice that this investigation never mentions the Law of Cosines by name. Rather than immediately ask students to remember the algebraic formula $c^2 = a^2 + b^2 - 2ab\cos C$, we think it makes sense to stick with geometric dissections, at least for a little while. In this way, students will view right triangles as a case in which two squares are added together to form a larger square and general triangles as a case in which two squares and *two parallelograms* are combined to form a larger square. This helps to emphasize that the Pythagorean Theorem needs only slight modifications in order for it to work for all triangles.

ASSESSMENT AND HOMEWORK IDEAS.......

A good way to assess students' understanding of this material is to ask them to present the geometry software construction to their classmates and explain how it leads to the results of Theorem 4.13.

WITHOUT TECHNOLOGY ....

This investigation can be done without geometry software, but it loses much of its impact because students are not able to experiment with different triangles. Even if you do not have enough computers for your whole class, it would still be effective if you used just one computer in the front of the classroom to demonstrate the construction.

MATHEMATICS CONNECTIONS ....

We chose not to equate this geometry software construction with the Law of Cosines, but if you're using this with a more advanced group, you can make the connection easily. Theorem 4.13 says that $c^2 = a^2 + b^2$ plus the area of two parallelograms. The area of these parallelograms combined is $2ab\sin(\theta - 90^\circ)$. From trigonometry, we know that $\sin(\theta - 90^\circ)$ is equal to $\sin\theta\cos 90^\circ - \cos\theta\sin 90^\circ = -\cos\theta$. Thus, $c^2 = a^2 + b^2 + 2ab(-\cos\theta)$, or equivalently, $c^2 = a^2 + b^2 - 2ab(\cos\theta)$.

Solutions

Investigation
4.1

Student Pages 1–4

INTRODUCTION TO MAPS AND BLUEPRINTS

Problem 1 *(Student page 1)*

The scale says that you can walk 600 feet in three minutes or 200 feet in one minute. Measure your own walking pace to see if this is reasonable for you.

Problem 2 *(Student page 1)*

a. The distance is approximately 1800 feet.

b. The distance is approximately 3900 feet.

Problem 3 *(Student page 1)* Lorena's walk is approximately 3900 feet and, according to the scale, will take nearly 20 minutes.

Problem 4 *(Student page 3)*

a. The left-hand chimney is about 30 feet tall, while the right-hand chimney is about 27 feet tall.

b. The garage door is about 10.5 feet wide by 7 feet tall. All of these measurements were obtained by using the known distance of 3 feet.

Problem 5 *(Student page 4)* The length of the unknown segment is about 162 inches.

Problem 6 *(Student page 4)* The overall dimensions are about 351 inches by 360 inches.

WHAT IS A SCALE FACTOR?

Problem 1 *(Student page 5)* See the explanations of Carlo and Amy in the Student Module for two possible ways to interpret this instruction.

Problem 2 *(Student page 7)* When you scale a square by a factor of $\frac{1}{2}$, the resulting figure is still a square, so it has all the properties shared by squares (equal sidelengths, all angles 90°, and so on). The numerical values of the sides, area, and perimeter all change, though. The side lengths are $\frac{1}{2}$ as long, the area is $\frac{1}{4}$ as large, and the perimeter is $\frac{1}{2}$ the original perimeter.

Problem 3 *(Student page 7)* The sides of the scaled square will be 3, 4, 8, 24, 12, and 15.6 inches, respectively.

Problem 4 *(Student page 7)* Scaling a figure by a factor less than 1 produces a smaller figure, while scaling by a factor greater than 1 enlarges it. Scaling by a factor of exactly 1 results in no change at all.

Problem 5 *(Student page 8)* The original picture has been scaled by a factor of $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

Problem 6 *(Student page 8)* The scale factors are $\frac{1}{3}$, $\frac{4}{5}$, $\frac{3}{5}$, and 1, respectively.

Because it is difficult to measure lengths in the figures accurately, your answers for parts b and c may be fractions whose values vary slightly from the answers given here.

Problems 7–8 *(Student page 9)* The scale factors are now the reciprocals of those given above: 3, $\frac{5}{4}$, $\frac{5}{3}$, and 1.

Problem 9 *(Student page 9)*

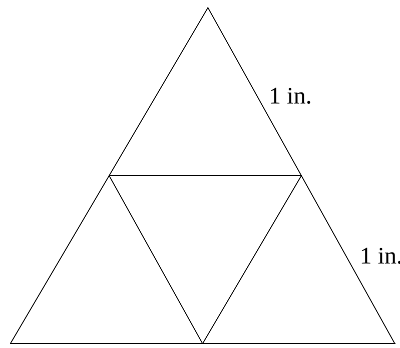
a. If 80% is entered on the machine, each dimension of the photocopy will be 80% as large as the original, giving a scaling factor of $\frac{80}{100} = \frac{4}{5}$.

b. To scale a picture by a factor of $\frac{3}{4}$, enter 75% on the machine.

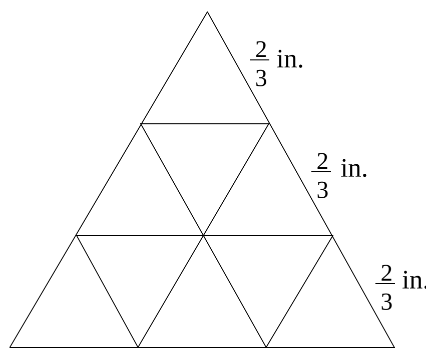
Problem 10 *(Student page 10)* The first two answers are 4 and 9. In general, if the 1-inch square is scaled by a positive integer r, then r^2 copies of it will fit inside the scaled version.

Problem 11 *(Student page 10)*

a. If you scale the triangle by $\frac{1}{2}$, you will get an equilateral triangle with sides of length 1 inch, and four of these triangles will fit inside the original.



b. If you scale the triangle by $\frac{1}{3}$, you will get an equilateral triangle with sides of length $\frac{2}{3}$ inch, and nine of them will fit inside the original.



Problem 12 *(Student page 10)* When a cube with edges of length 1 inch is scaled by factors of 2, 3, and r, its sides will become 2, 3, and r inches long, respectively. These scaled cubes have volumes 8, 27, and r^3 cubic inches, respectively. Since the original cube has volume 1 cubic inch, this means that 8, 27, and r^3 copies of it, respectively, will fit inside the scaled copy.

WORKING WITH DIRECTIONS: A LOGO ACTIVITY

Problems 1–2 *(Student page 12)* The Pent procedure contains three numbers: 5, 100, and 72. But changes in scale do not affect angles, so only the number that specifies *distance* in the procedure should be scaled.

Therefore, the half-sized pentagon is created by:

```
to Pent2
 repeat 5 [fd 50 rt 72]
end
```

If you try to draw a pentagon that is three times the size, it probably won't fit on the screen. But a procedure that will do it would look like this.

```
to Pent3
 repeat 5 [fd 300 rt 72]
end
```

In the module *Habits of Mind*, you saw the more general method for introducing variable elements. If you define a procedure like this

```
to VPent :scale
 repeat 5 [fd 100 * :scale rt 72]
end
```

you can then type `VPent 0.5` or `VPent 3` to solve this problem.

Exactly the same reasoning applies to modifying a procedure that draws your initials. Once you are successful drawing them at one scale, a change of scale requires you to multiply the new scale factor by *only* the distances used in the original procedure.

Problem 3 *(Student page 12)* There are many correct solutions to this task. You can start by drawing the outline of the house and then filling in the details. Or you can build "parts" of the house (like the front, roof, and door) separately and then assemble them. And, unlike a real building, there are many orders in which the parts

WORKING WITH DIRECTIONS: A LOGO ACTIVITY (continued)

can be assembled. The solution we present here will start by building the parts and assembling them beginning with, of all things, the door!

```
To Roof
 repeat 3 [forward 100   right 120]
end

To Front
 repeat 4 [forward 100  right 90]
end

To Door
 repeat 2 [fd 50  rt 90    fd 25  rt 90]
end

To House
 door
 left 90   forward 50   right 90
 front
 forward 100  right 30
 roof
 lt 30  back 100
 lt 90  back 50   rt 90
end
```

To draw the house, you need to clear Logo's screen and type the command House. It will draw the door first, then move the turtle to the left far enough to begin drawing the front of the house in the right place, and then move the turtle up to draw the roof. The last two lines of instructions in the house procedure—the lt 30 bk 100 and lt 90 bk 50 rt 90—are actually not needed to make the drawing. They put the turtle back where it began. (That is a bit of "neatness" that makes it easier to draw a row of houses if one wishes.)

Problems 4–6 *(Student pages 12–13)* As with the Pent and initials procedures, scaling the house requires multiplication of the scale factor by *all* of the lengths used in making the house. In our case, this means all the lengths in *each* of the procedures.

WORKING WITH DIRECTIONS: A LOGO ACTIVITY (continued)

So, to double the scale, one might modify the procedures this way:

```
to Roof
 repeat 3 [forward 100*2   right 120]
end


    ⋮


to House
 door
 left 90   forward 50*2   right 90
 front
 fd 100*2  rt 30
 roof
 lt 30  bk 100*2
 lt 90  bk 50*2   rt 90
end
```

More generally, one could write:

```
to Roof :scale
 repeat 3 [forward 100*:scale   right 120]
end

to Front :scale
 repeat 4 [forward 100*:scale  right 90]
end

to Door :sf
 repeat 2 [fd 50*:sf  rt 90   fd 25*:sf  rt 90]
end

to House :scale
 door :scale
 left 90   forward 50*:scale   right 90
 front :scale
 forward 100*:scale  right 30
 roof :scale
 lt 30  back 100*:scale
 lt 90  back 50*:scale   rt 90
end
```

WORKING WITH DIRECTIONS: A LOGO ACTIVITY (continued)

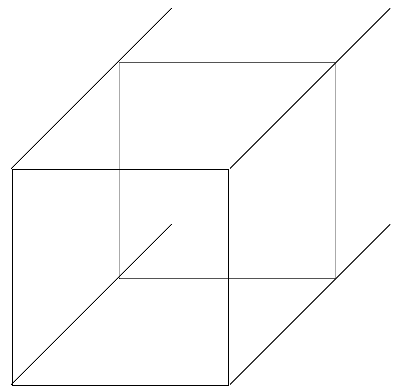
To use this set of procedures to draw the house at half-size, type **House 0.5** or **House 1/2**.

Problem 7 *(Student page 13)* This is where creating **Front** as its own procedure comes in handy! It can be recycled and used for the cube. Again, there are many solutions. Here's one:

```
To Front
 repeat 4 [forward 100  right 90]
end

To One.stick
 fd 50 * sqrt 2
 bk 50 * sqrt 2
end

To Four.sticks
 right 45  one.stick  left 45
 fd 100
 right 45  one.stick  left 45
 rt 90   fd 100
 left 45  one.stick  right 45
 rt 90   fd 100
 left 135  one.stick  right 135
 rt 90   fd 100
 rt 90
end

To Cube
 four.sticks
 rt 45
 fd 50 * sqrt 2
 lt 45
 front
end
```

This solution did not "clean up" at the end by moving the turtle back to its starting position.

Problem 8 *(Student page 13)* *Adding* 20 to each of the lengths makes the squares only 20% longer, but makes the diagonal connectors 40% longer.



Problems 9–11 *(Student pages 13–14)* As before, one must multiply by the scale factor. A general solution might look like this:

```
to Front :scale
 repeat 4 [forward 100 * :scale  right 90]
end

to One.stick :scale
 fd 50 * :scale * sqrt 2
 bk 50 * :scale * sqrt 2
end

to Four.sticks :scale
 right 45  one.stick :scale   left 45
 fd 100 * :scale
 right 45  one.stick :scale   left 45
 rt 90   fd 100 * :scale
 left 45  one.stick :scale   right 45
 rt 90   fd 100 * :scale
 left 135  one.stick :scale   right 135
 rt 90   fd 100 * :scale
 rt 90
end
```

```
to Cube :scale
 four.sticks :scale
 rt 45
 fd 50 * :scale * sqrt 2
 lt 45
 front :scale
end
```

After clearing the screen, try the sequence `cube 0.5 cube 0.75.` Does it look like the bigger cube is behind the smaller one? Why?

Problem 12 *(Student page 14)* `Pent1` is 0.618 times the size of `Pent0`. That is, 0.618 is the scale factor. Viewed from the other perspective, `Pent0` is $\frac{1}{0.618}$ the size of `Pent1`. Remarkably, $\frac{1}{0.618} \approx 1.618$. One must scale `Pent0` down twice by a factor of 0.618 to get `Pent2`, and so `Pent2` is 0.618^2 the size of `Pent0`, and `Pent0` is $1.618^2 \approx 2.618$ times the size of `Pent2`.

Problem 13 *(Student page 15)* The result is the triangular lace shown above Problem 12 in the Student Module.

Problem 14 *(Student page 15)*

a. The drawings made by the three procedures are so tiny that it is probably necessary to figure out what they are doing: the results are too small to see! `Bud` draws nothing (but may leave a single "point" to indicate where nothing was drawn!). `Tree1` draws a tiny twig with a 5° bend halfway along it. `Tree2` draws a stick similar to the `Tree1` twig, but at twice the size (scale factor 2).

b. When `Tree2` is edited, it draws a stick twice the size of the `Tree1` twig, but this time it has three `Tree1` twigs attached to it. (See a more detailed description in the Student Module just before Problem 15).

Problem 15 *(Student page 18)*

a. Five limbs grow from the trunk. The lowest limb sprouts 35% of the way up the trunk, branches off 18° to the left of the trunk, and grows to half the size of the trunk. The highest limb sprouts straight up from the top of the trunk; it, too, grows to half the length of the trunk.

b. Only three more instruction lines are needed. Here is the full procedure:

```
to Vtree4  :trunksize
 fd 0.35  * :trunksize
   lt 18   branch 0.5 * :trunksize   rt 18
 fd 0.15  * :trunksize
   rt 16   branch 0.4 * :trunksize   lt 16
 fd 0.3   * :trunksize
   lt 20   branch 0.35 * :trunksize   rt 20
 fd 0.12   * :trunksize
   rt 19   branch 0.3 * :trunksize   lt 19
 fd 0.08   * :trunksize
   lt 0   branch 0.5 * :trunksize   rt 0
 bk :trunksize
end
```

Notice how "35% of the way up the trunk," "18° to the left," and "grows to half the size of the trunk" are represented in the first two lines of the procedure.

The `rt 18` at the end of the second line faces the turtle the correct way, so that it can continue up the trunk. The last line, `bk :trunksize`, is essential to bring the turtle back to where it was when it started drawing the tree. If it were not there, then when `Vtree3` is substituted (later, in Problem 16) for `Branch`, the turtle would not be back at the base ("axil") of the limb, ready to turn back to continue along the trunk.

c. `Branch 10` draws a stick, 10 steps long; `branch 50` draws a stick 50 steps long. The pictures drawn by `Vtree4` are all similar. The input just determines the size (scale).

Problem 17 *(Student page 22)*

a. The circle centered at A has radius d_2. Thus, $\overline{AG}$ has length d_2. $\overline{GD}$ is a side of a small pentagon and, therefore, has length s_2. $AD - AG = GD$, implying $d_1 - d_2 = s_2$.

b. $s_1{}^2 = s_2 d_1$

c. $$\frac{s_1{}^2}{d_1{}^2} = \frac{s_2 d_1}{d_1{}^2}$$

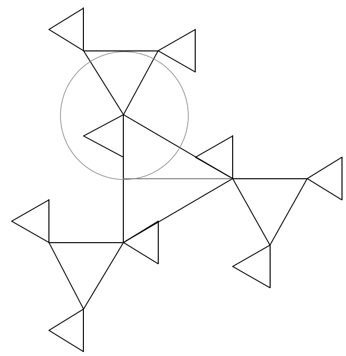
$$r^2 = \frac{s_2}{d_1}$$

d. $$r^2 = \frac{s_2}{d_1} = \frac{d_1 - d_2}{d_1} = \frac{d_1 - r d_1}{d_1} = \frac{d_1(1-r)}{d_1} = 1 - r$$

Problem 18 *(Student page 23)* The scale factor to use is $0.618\ldots^2 = 0.3819241\ldots$. For drawing purposes, 0.382 is good enough. One must nest procedures of this form:

```
To Pent5
 repeat 5 [fd 60  pent4  rt 72]
end

To Pent4
 repeat 5 [fd 60 * 0.382  pent3  rt 72]
end

To Pent3
 repeat 5 [fd 60 * 0.382 * 0.382  pent2  rt 72]
end

    ⋮
```

Problem 19 *(Student page 23)* You already know that the ratio used in the middle figure is between 0.5 and 0.65. One important number associated with equilateral triangles is $\sqrt{3}$, twice the altitude of an equilateral triangle of side 1. Playing with that number as a scale factor (something to multiply or divide by), one might notice that its reciprocal is $0.577\ldots$, just about in the right range. That ratio makes the smallest triangle in the figure on the next page exactly $\frac{1}{3}$ the scale of the largest and makes the altitude of the mid-sized triangle half the length of the side of the largest triangle. Does this prove that $\frac{1}{\sqrt{3}}$ is the correct ratio? No. The picture seems to verify the validity of this ratio, but none of this is proof. Still, looking at the geometry is a *very* effective way to find numbers that are "pleasing" scale factors.



Problem 20 *(Student page 24)* By replacing 0.33333 . . . with x, the statement

$$0.33333\ldots = 0.3 + \left(\frac{1}{10} \times 0.33333\ldots\right)$$

becomes

$$x = 0.3 + \left(\frac{x}{10}\right),$$

and the statement

$$10 \times 0.33333\ldots = 3.33333\ldots = 3 + 0.33333\ldots$$

becomes

$$10x = 3 + x.$$

Either of these can be solved, giving

$$x = \frac{1}{3}.$$

Problem 21 *(Student page 25)*

a. Many scale factors can be used, but the most obvious are 3 and 2 (or their reciprocals $\frac{1}{3}$ and $\frac{1}{2}$).

If you let

$$x = \frac{1}{3} + \frac{1}{9} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots,$$

then

$$3x = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{3^3} + \ldots = 1 + x,$$

or

$$\frac{x}{3} = \frac{1}{9} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \ldots = x - \frac{1}{3}.$$

b. Either of these (for example, $3x = 1 + x$) can be solved to show that $x = \frac{1}{2}$.

Similarly, multiplying the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ by 2 gives $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$. In other words, $2x = x + 1$, and so x must equal 1.

Problem 22 *(Student page 25)* It can appear nearly impossible to perform these calculations until you see how, within each calculation, a "small" copy of the same calculation is nested.

a. First name the continued fraction:

$$x = \frac{1}{1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}}}.$$

Then we can write

$$x = \frac{1}{1 + \left(\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}}\right)} = \frac{1}{1+x}.$$

The resulting equation,

$$x = \frac{1}{1+x},$$

can be rewritten as

$$x^2 + x - 1 = 0.$$

Solve this equation by the quadratic formula to get

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

$$x \approx -1.618 \quad \text{or} \quad x \approx 0.618.$$

Although the quadratic equation we have just solved has two solutions, you can see that the value of the continued fraction must be a positive number. Therefore, the value of the continued fraction is

$$\frac{-1+\sqrt{5}}{2} \approx 0.618.$$

b. If we let

$$x = \sqrt{1 + \left(\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}}}}\right)},$$

then we can write

$$x = \sqrt{1+x}.$$

Squaring both sides gives

$$x^2 = 1 + x$$

or

$$x^2 - x - 1 = 0.$$

Solve this equation by the quadratic formula to get

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$x \approx -0.168 \quad \text{or} \quad x \approx 1.618.$$

As in part a, you can see that, although the quadratic equation you have solved has two solutions, the value of the expression you were asked to evaluate must be a positive number. Therefore, the value of the continued radical is

$$\frac{1+\sqrt{5}}{2} \approx 1.618.$$

The solution to this problem is a special number called the "golden ratio" or "golden mean."

Solutions

Investigation **4.4**

Student Pages 26–29

What is a Well-Scaled Drawing?

Problem 1 *(Student page 27)* The third picture is the only well-scaled copy of the original. For most of the pictures, it is easy to tell that they are not scaled copies; they just don't look right. For instance, the skeleton may look too "stretched out," or the angle of the neck may have changed.

Problem 2 *(Student page 28)* The distances AB and EF correspond to each other, as do CD and GH. Thus, using the given measurements, you can compare their ratios:

$$\frac{AB}{EF} = \frac{CD}{GH} = 1.6$$

or

$$\frac{AB}{CD} = \frac{EF}{GH} = 2.$$

Either pair of proportions helps to establish that the two drawings are scaled copies. Of course, this doesn't show that *every* pair of corresponding measurements has the same ratio.

Problem 3 *(Student page 28)* When computing ratios, as long as the same unit is used for all measurements, the ratio will have no units attached to it. Suppose you have two segments $\overline{JK}$ and $\overline{LM}$ such that $JK = 5.2$ feet and $LM = 2$ feet. Then

$$\frac{JK}{LM} = \frac{5.2 \text{ feet}}{2 \text{ feet}} = 2.6.$$

We could also convert these measurements to inches and say that $JK = 62.4$ inches and $LM = 24$ inches. But still the ratio doesn't change:

$$\frac{JK}{LM} = \frac{62.4 \text{ inches}}{24 \text{ inches}} = 2.6.$$

So, when discussing ratios, there are no units. No matter what units of measurement we use in the example above, the ratio will always be 2.6.

Testing for Scale

Problem 1 *(Student page 30)* Computing the ratio of length to width for both rectangles shows that they are not scaled copies. You can also compare the ratio of the two rectangles' lengths to the ratio of their widths.

Problem 2 *(Student page 31)* The second and third pairs of rectangles are scaled copies.

What is the purpose of the dashed diagonal lines? They pass through the four pairs of corresponding rectangle vertices. They all come together at a single point if and only if the rectangles are scaled copies. You'll learn more about this in the upcoming investigations on dilation.

For Discussion *(Student page 32)* To check if two rectangles are scaled copies, you just need to see whether the ratio of their lengths equals the ratio of their widths. In this case, they do since $\frac{2}{3} = \frac{6}{9}$.

Another way to check the rectangles is to see whether the length-to-width ratio of one rectangle equals the length-to-width ratio of the other. Yes, $\frac{2}{6} = \frac{3}{9}$.

It's important to be able to use the terms "corresponding sides" and "proportional" properly, as they make talking about scaled copies much simpler. In the rectangles on *Student Module* page 32, the shorter sides of lengths 2 and 3 are corresponding sides, as are the sides of lengths 6 and 9. You can say that either the ratios of these *corresponding* sides are equal or the corresponding sides are *proportional*.

Problem 3 *(Student page 32)* There are three pairs of scaled copies. The $10'' \times 5''$ and the $8'' \times 4''$ rectangles are scaled copies (since $\frac{10}{5} = \frac{8}{4}$), as are the $4'' \times 1''$ and the $16'' \times 4''$ rectangles (since $\frac{4}{1} = \frac{16}{4}$).

The $3'' \times 2''$ and the $4'' \times 6''$ rectangles are also scaled copies. Notice, however, that $\frac{3}{2} \neq \frac{4}{6}$. To obtain equal ratios, you need to make sure your ratios are consistent. If one ratio is in the form "longer side/shorter side," then so must the other. Thus, we have the equal ratios $\frac{3}{2} = \frac{6}{4}$.

Problem 4 *(Student page 33)*

a. If the length of the scaled copy is l, then $\frac{l}{6} = 1.5$, so $l = 9$.

b. If the two rectangles have dimensions $l_1 \times w_1$ and $l_2 \times w_2$, then we know that

$$\frac{l_1}{l_2} = \frac{w_1}{w_2}$$

since they are scaled copies.

We are given that $\frac{l_1}{l_2} = 0.6$. Because this fraction is less than one, we know that $l_1 < l_2$. Thus, l_1 is the length of the smaller rectangle and $w_1 = 3$. We have

$$0.6 = \frac{3}{w_2},$$

implying that $w_2 = 5$.

When measuring these lengths and angles, as well as any others in the Student Module, don't worry if your ratios are slightly off. For example, if you compute the ratios of the corresponding sidelengths of two triangles and get values of 1.66, 1.8, and 1.6, you can be fairly certain that the triangles are drawn to scale.

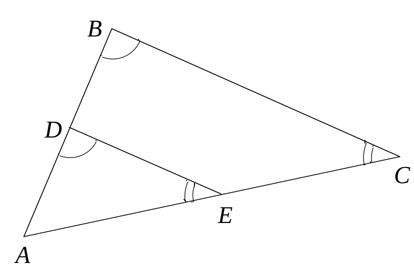
Problem 5 *(Student page 33)* These triangles are scaled copies. Notice that before comparing measurements of sides and angles, you must determine which sides and angles actually correspond. This is usually done by just looking at the pictures (the longest sides of each triangle correspond to each other, as do the shortest sides and the "middle-size" sides).

Problem 6 *(Student page 34)* Kaori is not looking at the proper ratios. The two smallest sides, the two longest sides, and the two "middle-size" sides are the pairs that go together. So the correct ratios to compare are $\frac{4}{6}$, $\frac{6}{9}$, and $\frac{8}{12}$. Since

$$\frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{2}{3} \approx 0.667,$$

this shows that the triangles are, in fact, scaled copies because you're already given that their corresponding angles are congruent.

Problem 7 *(Student page 34)* One way to check if triangles are scaled copies is to compare corresponding angles. Even though measuring is not allowed here, you can overlap the triangles to see if their angles match up. If two triangles $\triangle ADE$ and $\triangle ABC$ are scaled copies, then, when you lay one on top of the other, you'll get something like this:



TESTING FOR SCALE *(continued)*

Notice that $\triangle ADE$ and $\triangle ABC$ match perfectly at $\angle A$ and $\overline{DE}$ is parallel to side $\overline{BC}$.

Here are two possible conjectures about visual attributes of scaled triangles:

- All corresponding angles can be matched to coincide, one on top of the other.
- The triangles can be overlapped so that they coincide exactly at a common vertex, and the sides opposite this vertex are parallel.

Problem 8 *(Student page 36)* These triangles are not scaled copies.

Problem 9 *(Student page 37)* Some different approaches for determining if two triangles are scaled copies include:

- Check if corresponding angles have the same measurements and corresponding sides are proportional.
- Check if each side of one triangle has length r times the corresponding side length of the other triangle, for some positive number r.
- Compute the ratios of the lengths of corresponding sides to see if they are all the same.
- Measure corresponding angles to see if they are congruent.
- Place one triangle on top of the other to see if they line up as shown in the figure for Problem 7.
- Study the two triangles visually. If they are not scaled copies, it is often easy to see this without measuring. (They just don't look the same.)

Problem 10 *(Student page 37)*

Recall that corresponding angles formed when parallel lines are cut by a transversal are congruent.

a. It is, in fact, true that if corresponding angles of two triangles are congruent, then the triangles are scaled copies of each other. This will be proved later.

b. Because $\overline{DE}$ is parallel to $\overline{BC}$, it follows that

$$m\angle ADE = m\angle ABC$$

and

$$m\angle AED = m\angle ACB.$$

Since $m\angle A$ certainly equals $m\angle A$, we know that triangles $\triangle ABC$ and $\triangle ADE$ have congruent corresponding angles. The conjecture from Part a of this problem says that this suffices to conclude that $\triangle ADE$ is a scaled copy of $\triangle ABC$.

TESTING FOR SCALE *(continued)*

c. If this were true, then all rectangles would be scaled copies of each other, since they all have four right angles!

Problem 11 *(Student page 37)* This method doesn't work. What does work is to *multiply* each side by some fixed constant value. That's the definition of scaling.

Problem 12 *(Student page 38)* The left and middle triangles are scaled copies.

Problem 13 *(Student page 38)* The triangles are not scaled copies. Since you know that the sum of the angles in any triangle is 180°, you can calculate the measure of the third angle in each triangle to verify that the triangles do not have the same three angle measures.

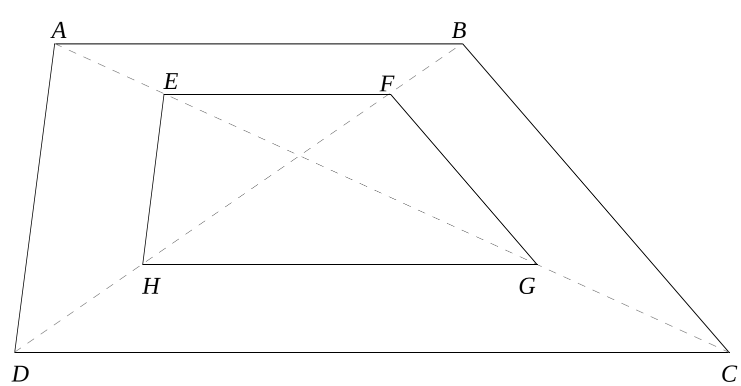
Problem 14 *(Student page 38)* These triangles are not scaled copies because the ratios $\frac{15}{12}$, $\frac{18}{14}$, and $\frac{21}{16}$ are not equal.

Problem 15 *(Student page 39)* Only the first pair of polygons are scaled copies. When dealing with arbitrary polygons, the fact that all corresponding angle measures are equal does not imply that the figures are scaled copies—you need to also check the ratios of corresponding sides.

This is probably most easily done with geometry software.

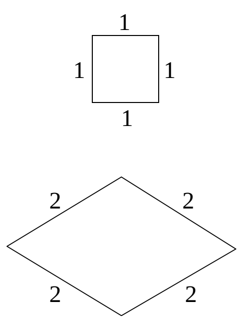
Problem 16 *(Student page 40)* Here's one interesting approach that relates back to the dashed lines in Problem 2: First, draw two line segments connecting the opposite vertices of the trapezoid. Then construct a new trapezoid, each of whose vertices lie on these lines, with sides parallel to the sides of the original trapezoid. This new trapezoid will be a scaled copy of the original.

You will explore this method in depth in the upcoming dilation investigations.



Is there another way to solve this problem that doesn't require the dashed lines?

You may have noticed by now that scaled polygons can be oriented so that their corresponding sides are parallel. Using this fact, you can solve the problem a bit differently. Instead of drawing dashed diagonals, begin by drawing side $\overline{EF}$ inside of $ABCD$, parallel to $\overline{AB}$, with any length you like. Then, by measuring the lengths of $\overline{EF}$ and $\overline{AB}$, calculate the scaling factor, r. Now draw in the remaining three sides of your new polygon, $\overline{FG}$, $\overline{GH}$, and $\overline{HE}$, with the properties that each one is parallel to its corresponding side in $ABCD$, and each has the length necessary to ensure that all pairs of corresponding sides have the same scaling factor, r.



Problem 17 *(Student page 40)* An example is a square and a nonsquare rhombus, where the sides of one are twice as long as the sides of the other (see the picture in the margin). They are not scaled copies since their angles differ.

Problem 18 *(Student page 40)* Neither statement is accurate. For the first one, recall that all rectangles have congruent corresponding angles, but not all are scaled copies of each other. The example given in the solution to Problem 17 provides a counterexample to the second test.

What *is* true, however, is that if two polygons have corresponding angles with the same measures, *and* if their corresponding sides have the same ratio, then the two polygons are scaled copies.

Problem 19 *(Student page 40)*

a. Suppose a side of the original polygon has length a cm. Then the corresponding side of the scaled copy has length $\frac{3}{4}a$ cm. If you look at their ratio, you get

$$\frac{\frac{3}{4}a}{a} = \frac{3}{4}$$

or

$$\frac{a}{\frac{3}{4}a} = \frac{4}{3},$$

depending on which way you set up the ratio. Notice that $\frac{3}{4}$ and $\frac{4}{3}$ are reciprocals.

b. The ratio of any two corresponding angles of the polygon will be 1, as scaling preserves angle measurement.

TESTING FOR SCALE *(continued)*

Problem 20 *(Student page 41)*

a. Take any square and any trapezoid; they are both quadrilaterals but are definitely not scaled copies, as they don't have the same essential shape.

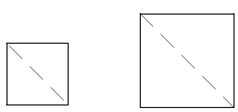
b. Any two squares are scaled copies. Suppose you have one square with sides of length a cm and another with sides of length b cm. Since they are squares, all their angles measure 90°. Each side of the second square is $\frac{b}{a}$ as long as each side of the first square. Hence, they are scaled copies with a scaling factor of $\frac{b}{a}$.

c. No; remember again the example of a rectangle and a square. They both consist of all right angles but are not necessarily scaled copies.

We will prove later that if two triangles have the same angle measurements, then they are scaled copies. This is a special fact about triangles.

d. No, any two triangles are not automatically scaled copies. Consider a right triangle and a nonright triangle; they are clearly not scaled copies as they have completely different shapes.

e. Suppose you have an isosceles right triangle and a nonright triangle that is also isosceles. They're not scaled copies since they have different shapes.



f. Any two isosceles right triangles will be scaled copies. Any isosceles right triangle can be considered half of a square, as a square's diagonal divides it into two isosceles right triangles. So, given any two isosceles right triangles, join together two copies of each to form two squares. These squares will be scaled copies of each other (as are all squares), so the two triangles are also scaled copies.

g. Any two equilateral triangles are scaled copies. If one triangle has all three sides of length a and the other has all three sides of length b, then all three ratios of corresponding sides are equal to $\frac{a}{b}$, and all the angles measure 60°. Thus, the triangles are scaled copies.

h. Any two rhombuses are not necessarily scaled copies. If one rhombus is a square but the other is not, they won't have equal angle measurements.

i. Any two regular polygons with the same number of sides will be scaled copies. Regular polygons are both equiangular and equilateral. If two regular polygons have the same number of sides, then all angle measurements will be equal. Moreover, if all the sides of one have length a and all the sides of the other have length b, then all ratios of correspondings sides will be equal to $\frac{a}{b}$. Thus, the two polygons will be scaled copies.

The Many Faces of Scaling

Problem 1 *(Student page 43)* Student drawings for these problems will vary.

RECTANGLE DIAGONALS

Problems 1–2 *(Student pages 44–45)* In this experiment, you discover that scaled rectangles are precisely those whose diagonals line up with each other when corresponding corners of the rectangles are aligned. This gives a method for testing whether two rectangles are scaled copies without having to take any measurements.

Problem 3 *(Student page 45)* Let rectangle 1 be the original $8\frac{1}{2}'' \times 11''$ rectangle. Then let rectangle 2 be the rectangle obtained after Step 1 by tearing rectangle 1 in half. Let rectangle 3 be the one you get by folding and tearing rectangle 2 in half, and continue numbering them in this way. All the odd-numbered rectangles are scaled copies of each other, and all the even-numbered rectangles are also scaled copies of each other.

To see what is happening, suppose rectangle 1 has dimensions $a \times b$. We can then list the dimensions of the first few rectangles:

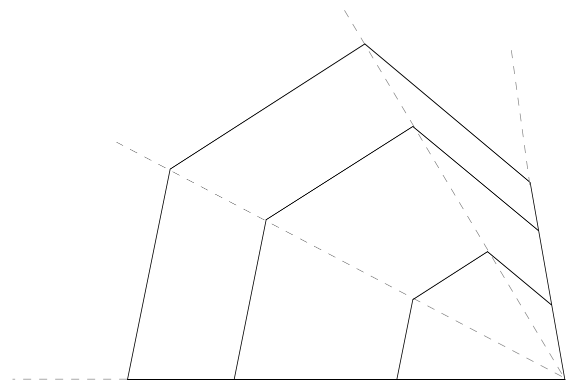
$$\begin{array}{ccc} a & \times & b \\ a & \times & \frac{b}{2} \\ \frac{a}{2} & \times & \frac{b}{2} \\ \frac{a}{2} & \times & \frac{b}{4} \\ \frac{a}{4} & \times & \frac{b}{4} \\ \frac{a}{4} & \times & \frac{b}{8} \end{array}$$

Notice that, if you pick any rectangle, the one two steps below it on the list will have both length and width exactly half as big; hence these two rectangles will be scaled copies, with a scaling factor of $\frac{1}{2}$. Therefore, all even-numbered rectangles are scaled copies of each other, as are all odd-numbered rectangles.

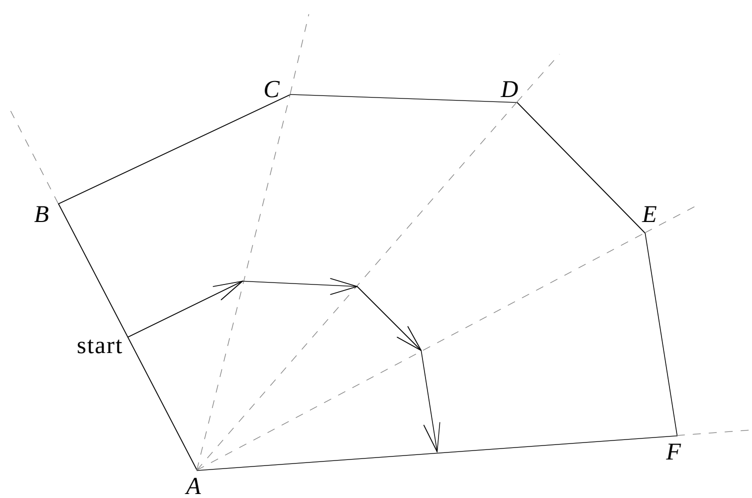
There is nothing special about the original $8\frac{1}{2}'' \times 11''$ measurements here; the same will be true no matter what the size of the first rectangle.

RECTANGLE DIAGONALS (continued)

Problem 4 *(Student page 45)* The diagonals and sides of the polygons will line up, as in the picture below:

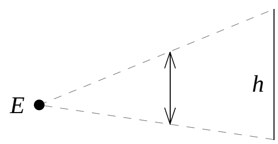


Problem 5 *(Student page 46)* The following method for scaling a polygon is suggested by the results of the previous problem. Begin by finding the midpoint of one side of *ABCDEF*, say $\overline{AB}$. Then starting at that midpoint, draw in line segments between the dashed lines that are parallel to each corresponding side of *ABCDEF*.



You may notice that each vertex of the scaled polygon is exactly halfway between *A* and the corresponding vertex of *ABCDEF*. This suggests another way to construct the scaled copy: find the midpoint of each dashed line segment, and then connect consecutive midpoints. You will explore this method later in this module.

LIGHT AND SHADOWS: PROJECTED IMAGES



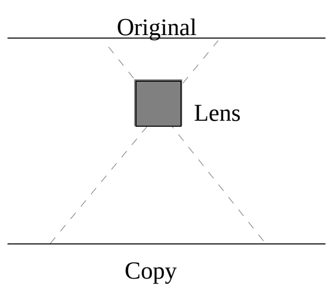
Problems 1–2 *(Student page 48)* As you move your fingers closer to your eyes, you need to bring them closer together, narrowing the bracket they make. In the margin picture, point E represents your eye, h represents the height of the object you are viewing, and the distance between the dashed lines represents the size of the bracket made by your fingers at various distances from your eye. The dashed lines represent the lines of sight from your eye to the top and bottom of the object.

Problem 3 *(Student page 49)* The image on the screen will become larger if you move the projector away from the screen, and will become smaller if you move it closer to the screen. To distort the image, tilt the projector or screen at an angle.

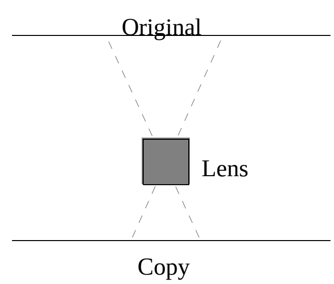
Problem 4 *(Student page 50)* The closer an object is to the light source, the larger its shadow will be. For example, suppose the distance from a light source to a wall is d. If the distance from the light to the object is $\frac{1}{2}d$, the shadow cast on the wall will be approximately twice as large as the object. If the distance from the light to the object is $\frac{1}{3}d$, the shadow will be approximately three times as large.

Problem 5 *(Student page 50)* To cast a 17″ × 22″ shadow, place the paper exactly halfway between the wall and the light source. The paper would have to be placed practically on the wall in order to cast a same-size shadow.

Problem 6 *(Student page 50)* To make a copy the same size as the original, the lens should be halfway between the copier glass and the screen. If the lens is closer to the glass than to the screen, the copy will be larger than the original. If the lens is closer to the screen than to the glass, the copy will be smaller than the original. The amount by which the original is enlarged or reduced depends on the exact distance of the lens from the glass and screen.



Enlargement

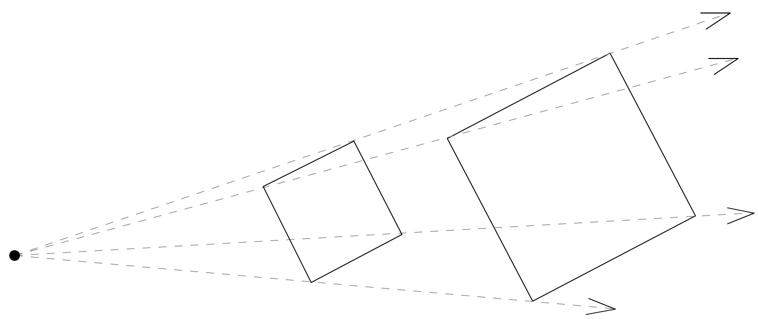


Reduction

Curved or Straight? Just Dilate!

Problem 1 *(Student page 52)* No matter what point is chosen as the center of dilation, the resulting figure will be a copy of the circle scaled by a factor of 2. By changing the center of dilation, the location of the scaled copy on the page changes, but that is all.

Problem 2 *(Student page 52)* The orientation of the new square will be exactly the same as that of the original tilted square:



Problem 3 *(Student page 52)* Suppose point C is the center of dilation. Then, if point A is on the figure, construct the ray through C and A as before. This time, however, the point A' that you construct on the ray should have the property that it is half as far from C as A is from C. In other words, $CA' = \frac{1}{2}CA$. To scale by a factor of 3, choose A' so that $CA' = 3CA$. Do this for a variety of points, and then connect them to obtain your scaled figure.

Problem 4 *(Student page 52)* To find the center of dilation, first locate several pairs of corresponding points on the two figures. Then draw a line through each pair of points. You'll see that all the lines share a common point of intersection, which is the center of dilation.

Problem 5 *(Student page 53)* Unless your students are extremely artistic, don't expect them to draw wonderfully scaled pictures! The idea is to pick a few essential features to dilate—ears, nose, eye, and so on.

Problem 6 *(Student pages 53–54)*

a. The paths traced by points B and M are scaled copies of each other, with B's path larger by a factor of 2. Think of point A as the center of dilation: as point B is dragged around, A stays fixed and B is always twice as far from A as M is from A.

b. When point B is moved along the sides of a polygon, point M traces out a smaller copy of the polygon, scaled by a factor of $\frac{1}{2}$. Each point on the polygon traced by M is $\frac{1}{2}$ as far from A as the corresponding point on the polygon traced by B.

c. When the entire $\overline{AB}$ is traced, you see the figure traced by B, the scaled copy traced by M, and the rays (in this case, segments) used to make the dilation.

Problem 8 *(Student page 54)* The measurements on the mirror should be close to half the measurements of your face.

Problem 9 *(Student page 54)* Think of your eyes as the center of dilation. Imagine drawing lines from your eyes to the image “on the other side” of the mirror. The mirror itself is halfway in between, causing your image on the mirror to be half the size of your actual face.

Problem 10 *(Student page 54)* No, you will not get the same picture as before. Your friend will be tracing the image from a different center (her own eyes).

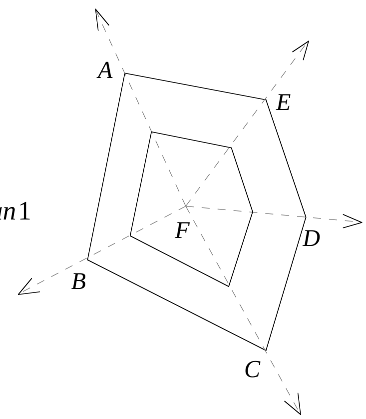
Ratio and Parallel Methods

Problem 1 *(Student page 56)* The new polygon is a scaled copy of the original, scaled by a factor of $\frac{1}{2}$. You can verify this by measuring all the sides and angles of each figure.

Problem 2 *(Student page 56)* When you use the ratio method to scale a polygon, the sides of the scaled polygon will be parallel to the corresponding sides of the original polygon. This is important and will actually be proved later on.

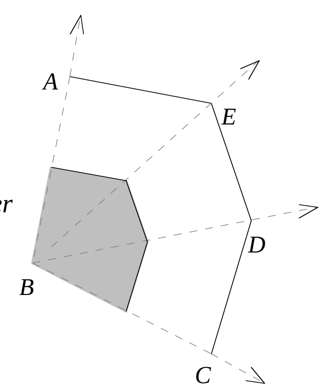
Problem 3 *(Student page 56)* If point F is inside the polygon, the ratio method still works. Now, if the dilation factor is less than 1, the smaller copy lies entirely inside the original:

F inside the polygon, dilation factor less than 1



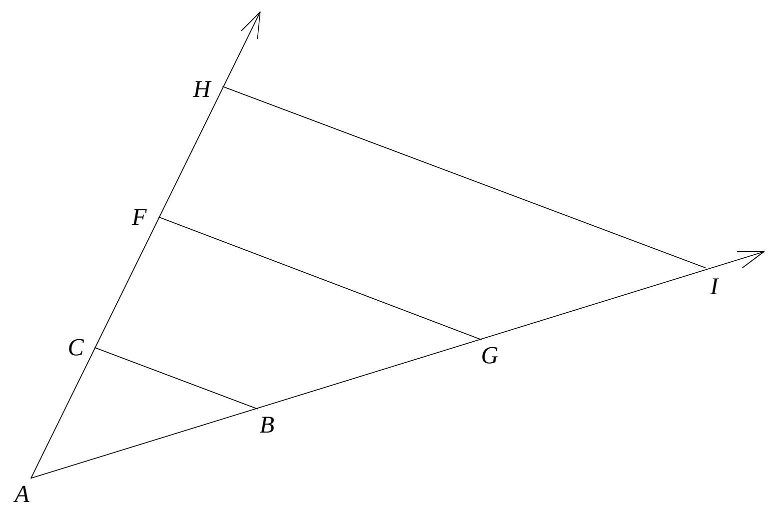
Problem 4 *(Student page 56)* If point F is at a vertex of $ABCDE$, the scaled copy will share a corner with the original. The illustration below shows what happens when point F is placed at vertex B; the shaded figure is the scaled copy.

B is the center of dilation.



Problems 5–6 *(Student page 56)* To dilate a polygon by $\frac{1}{3}$, construct rays through F and each vertex as before, but then find the point on each ray whose distance from F is $\frac{1}{3}$ the distance from the vertex to F. To dilate a polygon by 2, find the point on each ray that is twice as far from F as the polygon vertex.

Problem 7 *(Student page 56)* In the picture below, $\triangle AGF$ is a dilation of $\triangle ABC$ by a factor of 2, while $\triangle AIH$ is a dilation of $\triangle ABC$ by a factor of 3. Notice that $\overline{BC}$, $\overline{GF}$, and $\overline{IH}$ of the three triangles are all parallel.



The triangles were constructed so that

$$AF = 2AC$$
$$AG = 2AB,$$

and

$$AH = 3AC$$
$$AI = 3AB.$$

Problem 8 *(Student page 58)* Steve actually dilated the triangle by a factor of 3, not 2. In order for the new triangle to be a dilation by a factor of 2, the

following must be true:

$$\begin{aligned} DA' &= 2DA \\ DB' &= 2DB \\ DC' &= 2DC. \end{aligned}$$

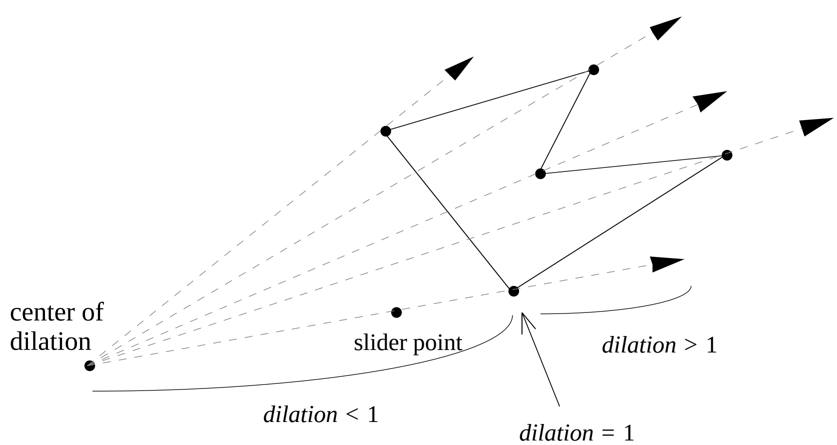
What was Steve's mistake? He chose point A' so that $DA' = 3DA$ and repeated this pattern with the other two triangle vertices.

Problem 9 *(Student page 58)* Triangle ABC was scaled by a factor of about 1.4. One way to see this is to draw in the rest of two of the rays in the picture; they will meet at the center of dilation, D. Then by measuring, you can see that $DA' = 1.4(DA)$. Alternatively, you can measure the ratio of any two corresponding sides of the triangles.

Problem 10 *(Student page 59)* You can measure each side and angle of both polygons to check that the polygons are scaled copies. Or, you can just check that each of the remaining three vertices of the copy are twice the distance from E as the corresponding vertex of $ABCD$. If this is the case, then the ratio method says that they are scaled copies.

Problem 11 *(Student page 59)* This problem provides more opportunities to practice the dilation method. To start, pick a point E for the center of dilation, and draw rays from E through each vertex. Then, if A is a vertex of the original polygon, construct a new point A' on the ray so that $EA' = \frac{1}{2}EA$ (or so that $EA' = 3EA$ in the second case). Then, starting at A', move around the rays, drawing segments parallel to the sides of the original polygon.

Problem 12 *(Student page 60)* See the figure below:



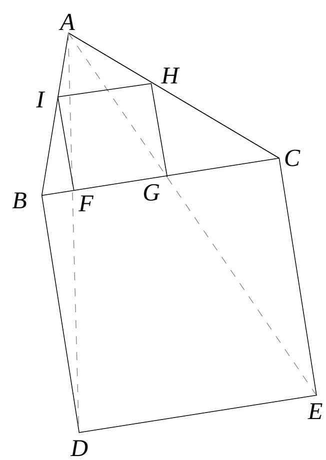
RATIO AND PARALLEL METHODS *(continued)*

Problem 13 *(Student page 60)* This problem is tricky. First, construct a square facing away from $\triangle ABC$ that has $\overline{BC}$ as a side. Label the other vertices of this square D and E.

Draw segments from A to each vertex of the large square ($\overline{AB}$ and $\overline{AC}$ already exist; you just need to add $\overline{AD}$ and $\overline{AE}$). The key to this problem is to think of point A as a center of dilation and to dilate square $BCED$ so that a side of it lands on $\overline{BC}$ of the triangle.

You're using the parallel method to dilate square *BCED*.

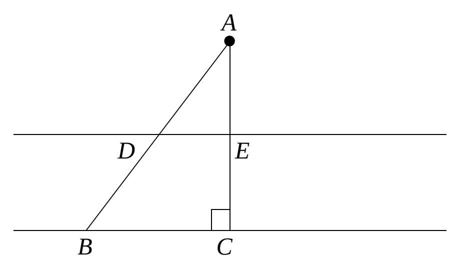
Let F be the intersection point of $\overline{AD}$ and $\overline{BC}$, and let G be the intersection point of $\overline{AE}$ and $\overline{BC}$. Now start at F and draw segments parallel to each side of square $BCED$. When you're done, you'll have drawn a square sitting inside of $\triangle ABC$.



NESTED TRIANGLES: BUILDING DILATED POLYGONS

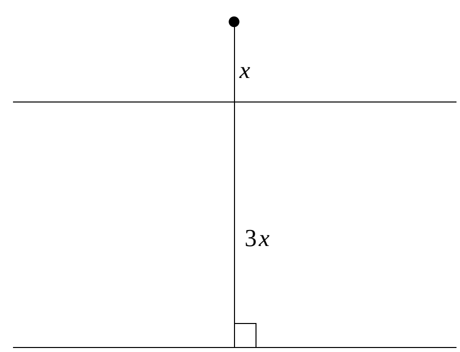
Problem 1 *(Student page 63)* Draw ten different line segments from the point to the bottom line. The midpoint of each segment is the point at which the segment crosses the top parallel line.

To see this, look at the picture below. $\overline{AC}$ is drawn perpendicular to the parallel lines, and $\overline{AB}$ is an arbitrary segment drawn between the point and the bottom line. We need to show that point D is the midpoint of $\overline{AB}$.



Since the point and the lines are evenly spaced, $AE = EC$; thus, $\frac{AE}{EC} = 1$. From the previous problem, you know that $\frac{AD}{DB} = \frac{AE}{AC}$. So $\frac{AD}{DB} = 1$, giving $AD = DB$.

Problem 2 *(Student page 63)* For this setup, draw the parallel lines so that the distance between the lines is three times the distance from the top line to the point.



Problem 3 *(Student page 64)* As point A is dragged along the entire length of $\overline{XY}$, point M will trace out a line segment that is parallel to $\overline{XY}$ and half its length.

Problem 4 *(Student page 64)* Point M still traces out a segment parallel to $\overline{XY}$, but now its length is equal to $\frac{BM}{BA}$ times the length of $\overline{XY}$.

Problem 5 *(Student page 64)* Each midpoint will trace out a line segment parallel to segment XY, and all of these segments will have one half the length of $\overline{XY}$.

This is similar to the Parallel Theorem that you'll see later.

Problem 6 *(Student page 65)* You will discover that the following ratios are all equal:

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$$

and also

$$\frac{AD}{DB} = \frac{AE}{EC}.$$

Problem 7 *(Student page 66)*

- Part a of the Parallel Theorem says that if $\overline{DE}$ is parallel to $\overline{BC}$, with D on $\overline{AB}$ and E on $\overline{AC}$, then
$$\frac{AD}{AB} = \frac{AE}{AC}$$
or, equivalently, that
$$\frac{AD}{DB} = \frac{AE}{EC}.$$
- Part b of the Parallel Theorem adds a third ratio to the first proportion given above for Part a. It says that, if $\overline{DE}$ is parallel to $\overline{BC}$, then
$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}.$$
- The Side-Splitting Theorem is the converse of part a of the Parallel Theorem:

  If
$$\frac{AD}{AB} = \frac{AE}{AC},$$
then $\overline{DE}$ is parallel to $\overline{BC}$.

Problem 8 *(Student page 67)* The phrase "part is to part as part is to part" refers to the fact that D and E each divide their respective sides into two parts, and these parts have the same ratio:

$$\frac{AD}{DB} = \frac{AE}{EC}.$$

The phrase "part is to whole as part is to whole" refers to the ratio of the top "part" of each side to the whole side:

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

Problem 9 *(Student page 67)* All of these lengths are obtained by applying the Parallel Theorem:

a. $AC = 6$

b. $AD = 16$

c. $EC = 8$

d. $BC = 12$

e. Since $AD = 2$ and $DB = 6$, it follows that $AB = 8$. Thus,

$$\frac{DE}{BC} = \frac{AD}{AB} = \frac{2}{8} = \frac{1}{4}.$$

Problem 10 *(Student page 68)* Notice that

$$\frac{r - s}{s} = \frac{r}{s} - 1$$

and

$$\frac{t - u}{u} = \frac{t}{u} - 1.$$

Since you are given that

$$\frac{r}{s} = \frac{t}{u},$$

it follows that these two quantities are equal.

Problem 11 *(Student page 68)* Assume that $\frac{AB}{AD} = \frac{AC}{AE}$.

Using the previous problem as a guide, you get:

$$\frac{AB - AD}{AD} = \frac{AC - AE}{AE},$$

which implies

$$\frac{DB}{AD} = \frac{EC}{AE}.$$

These steps can be retraced to show that if you assume $\frac{DB}{AD} = \frac{EC}{AE}$, then $\frac{AB}{AD} = \frac{AC}{AE}$.

Problem 12 *(Student page 68)* The Midline Theorem is a special case of the Side-Splitting Theorem because midlines split sides of triangles proportionally—into halves—so the Side-Splitting Theorem holds.

SIDE-SPLITTING AND PARALLEL THEOREMS

Problem 1 *(Student page 69)* As point C moves, the area of $\triangle ABC$ does not change at all. The area of a triangle is one half base times height. In this case, the base will always have length AB; this is independent of C's location. And the height will always be the perpendicular distance from C to the line containing $\overline{AB}$. Since C is on a line parallel to $\overline{AB}$, this distance will always be the same, regardless of C's location. Thus the area of $\triangle ABC$ does not depend on C, and so never varies as point C moves.

Problem 2 *(Student page 69)*

a. The area of $\triangle ABC$ is $\frac{1}{2}h(AC)$, while the area of $\triangle DEF$ is $\frac{1}{2}h(DF)$. The ratio of their areas is thus

$$\frac{\frac{1}{2}h(AC)}{\frac{1}{2}h(DF)} = \frac{AC}{DF}.$$

This shows that if two triangles have the same height, then the ratio of their areas is equal to the ratio of their bases.

b. Triangles $\triangle GEM$ and $\triangle MEO$ have the same height: the length of the altitude from M to $\overline{GO}$. Therefore, the value of $\frac{GE}{EO}$ is the same as the ratio of the areas of the triangles, which is $\frac{3}{2}$.

Now, triangles $\triangle GEM$ and $\triangle GOM$ also have the same height. We can apply the same reasoning as above to conclude that

$$\begin{aligned}\frac{GE}{GO} &= \frac{\text{Area }(\triangle GEM)}{\text{Area }(\triangle GOM)} \\ &= \frac{\text{Area }(\triangle GEM)}{\text{Area }(\triangle GEM) + \text{Area }(\triangle MEO)} \\ &= \frac{3}{3+2} \\ &= \frac{3}{5}.\end{aligned}$$

Problem 3 *(Student page 70)*

a. Triangles $\triangle ACB$ and $\triangle ADB$ share the same base, $\overline{AB}$. The height of $\triangle ACB$ is equal to the distance from C to $\overline{AB}$, and the height of $\triangle ADB$ equals the distance from D to $\overline{AB}$. But, since $\overline{DC}$ and $\overline{AB}$ are parallel, these two distances are the same. Thus, the triangles have equal bases and heights, and hence the same areas.

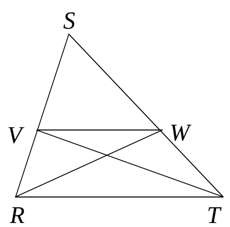
b. Triangles $\triangle ADC$ and $\triangle BCD$ also have equal areas, as they share base $\overline{DC}$ and have the same height.

Both $\triangle ADC$ and $\triangle BCD$ contain $\triangle DPC$. If you "remove" $\triangle DPC$ from both of these triangles, you're left with $\triangle ADP$ and $\triangle BCP$, which also have equal areas.

Problem 5 *(Student page 77)* This time we are given that

$$\frac{SV}{VR} = \frac{SW}{WT},$$

and asked to prove that $\overline{VW} \parallel \overline{RT}$.



From Euclid's proof, you know that $\triangle SVW$ and $\triangle RVW$ have the same height, as do $\triangle SVW$ and $\triangle TVW$. Thus,

$$\frac{SV}{VR} = \frac{\text{Area }(\triangle SVW)}{\text{Area }(\triangle RVW)}$$

and

$$\frac{SW}{WT} = \frac{\text{Area }(\triangle SVW)}{\text{Area }(\triangle TVW)}.$$

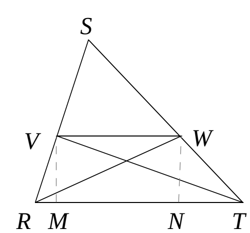
Using the given proportion, you get

$$\frac{\text{Area }(\triangle SVW)}{\text{Area }(\triangle RVW)} = \frac{\text{Area }(\triangle SVW)}{\text{Area }(\triangle TVW)},$$

which means that

$$\text{Area }(\triangle RVW) = \text{Area }(\triangle TVW).$$

Now, draw segments $\overline{VM}$ and $\overline{WN}$, intersecting $\overline{RT}$ perpendicularly.



SIDE-SPLITTING AND PARALLEL THEOREMS
(continued)

We can rewrite the area equality

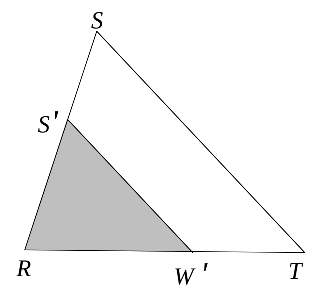
$$\frac{1}{2}(VW)(VM) = \frac{1}{2}(VW)(WN),$$

which implies that

$$VM = WN.$$

$\overline{VM}$ and $\overline{WN}$ are both perpendicular to $\overline{RT}$, and have the same length. It follows that $\overline{VW}$ is parallel to $\overline{RT}$.

Problems 6–7 *(Student page 78)* Because $\overline{VW} \parallel \overline{RT}$, you know that $m\angle SVW = m\angle SRT$, so $\triangle SVW$ slides right into the corner of $\triangle SRT$:



Relabel the new picture with points S' and W' as shown. Notice that now $\overline{S'W'} \parallel \overline{ST}$, since $m\angle RS'W' = m\angle RST$. Also notice that $\triangle RS'W'$ is an exact relabeled copy of $\triangle SVW$, so it follows that

$$\begin{aligned} S'W' &= SW \\ RS' &= VS \\ RW' &= VW. \end{aligned}$$

Problem 8 *(Student page 78)* Because $\overline{S'W'} \parallel \overline{ST}$, you can apply the Parallel Theorem to $\triangle RST$, yielding the equality

$$\frac{RT}{RW'} = \frac{SR}{RS'}.$$

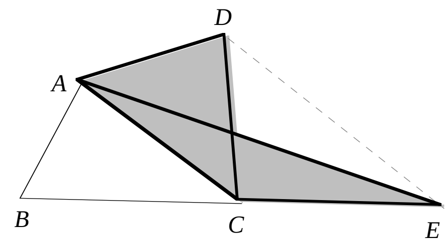
Substituting from the equalities listed in the last problem, you get

$$\frac{RT}{VW} = \frac{SR}{SV}.$$

Including the equality you already know, you can conclude that

$$\frac{RT}{VW} = \frac{SR}{SV} = \frac{ST}{SW}.$$

Problem 9 *(Student page 79)* To begin, $\triangle ADC$ has the same area as $\triangle ACE$.

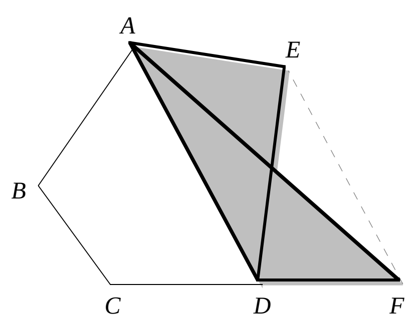


This follows because two triangles with the same base ($\overline{AC}$) and equal heights (the distance between $\overline{AC}$ and $\overline{DE}$) have the same area.

Now, break up the area of $\triangle ABE$:

$$\begin{aligned} \text{Area}\ (\triangle ABE) &= \text{Area}\ (\triangle ABC) + \text{Area}\ (\triangle ACE) \\ &= \text{Area}\ (\triangle ABC) + \text{Area}\ (\triangle ADC) \\ &= \text{Area}\ (ABCD). \end{aligned}$$

Problem 10 *(Student page 80)* Apply the technique from Problem 9 twice to pentagon $ABCDE$. First draw $\overline{AD}$ and then draw a line through E parallel to $\overline{AD}$, extending $\overline{CD}$ so that it meets this line at point F.

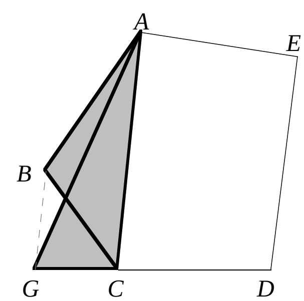


We get

$$\text{Area } (\triangle\, ADE) = \text{Area } (\triangle\, ADF)$$

because they both share base $\overline{AD}$ and have height equal to the distance between parallel segments EF and AD.

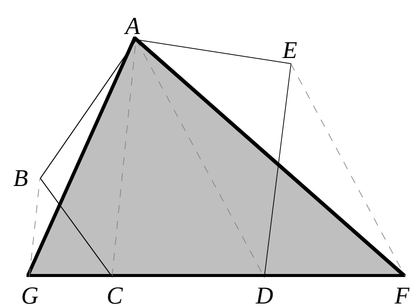
Now we repeat this process. This time draw $\overline{AC}$, and draw a line through B parallel to $\overline{AC}$, extending $\overline{CD}$ to meet this line at point G.



Notice that

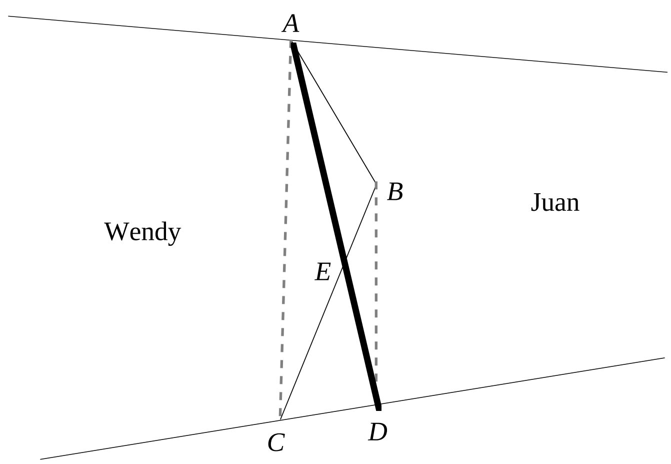
$$\text{Area } (\triangle\, ACG) = \text{Area } (\triangle\, ACB).$$

Finally, form $\triangle\, AGF$ and calculate its area, substituting from the area equations above.

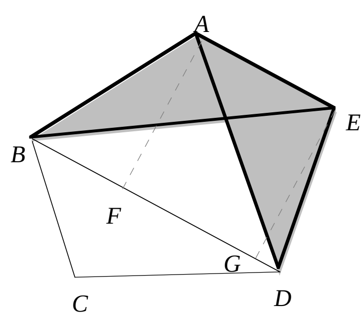


$$\begin{aligned}\text{Area } (\triangle\, AGF) &= \text{Area } (\triangle\, ACG) + \text{Area } (\triangle\, ACD) + \text{Area } (\triangle\, ADF)\\ &= \text{Area } (\triangle\, ACB) + \text{Area } (\triangle\, ACD) + \text{Area } (\triangle\, ADE)\\ &= \text{Area } (ABCDE)\end{aligned}$$

Problem 11 *(Student page 80)* Draw $\overline{AC}$, and then construct $\overline{BD}$ parallel to $\overline{AC}$ through B. Draw $\overline{AD}$. Using $\overline{AD}$ as the new border divides the land properly since area $(\triangle ABC)$ = area $(\triangle ADC)$. Both of these triangles have base AC, and the corresponding heights are both the distance between parallel segments BD and AC. $\triangle ABC$ was Wendy's land before the change and $\triangle ACD$ is her land after the change, so the area of Wendy's land has not changed.



Problem 12 *(Student page 81)* Why would it help to show that $\triangle ABE$ and $\triangle ADE$ have the same area? If they did, then their heights (AF and EG) would be equal since the triangles share the same base ($\overline{AE}$). As $\overline{AF}$ and $\overline{EG}$ are both perpendicular to $\overline{AE}$, this would imply that $\overline{EA} \parallel \overline{BD}$.



So how can you show that area $(\triangle ABE) =$ area $(\triangle ADE)$?

Use the given four pairs of parallel line segments to obtain four pairs of triangles, each having the same area.

The statements

$$\overline{AB} \parallel \overline{CE}$$
$$\overline{BC} \parallel \overline{AD}$$
$$\overline{CD} \parallel \overline{BE}$$

and

$$\overline{DE} \parallel \overline{CA}$$

imply the following area statements, respectively:

$$\begin{aligned}
\text{Area } (\triangle ABE) &= \text{Area } (\triangle ABC) \\
\text{Area } (\triangle ABC) &= \text{Area } (\triangle BCD) \\
\text{Area } (\triangle BCD) &= \text{Area } (\triangle CDE) \\
\text{Area } (\triangle CDE) &= \text{Area } (\triangle ADE).
\end{aligned}$$

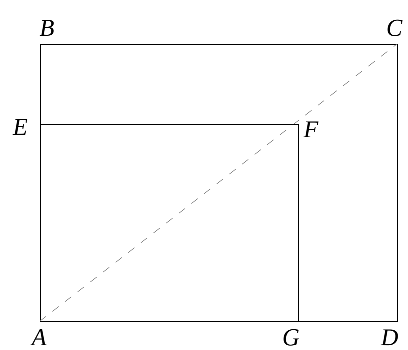
Something very nice happens here. You can trace through the equalities above and conclude that all of these triangles have the exact same area. In particular,

$$\text{Area } (\triangle ABE) = \text{Area } (\triangle ADE),$$

which completes the problem. Pretty nice!

Problem 13 *(Student page 81)*

a. Because the inner rectangle fits into the corner of the outer one, the corresponding sides of the two rectangles are parallel. Label the vertices of the rectangles as shown here:



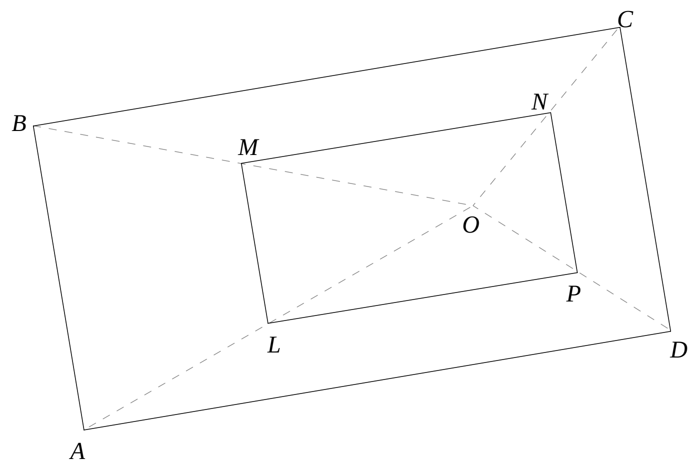
Now look at $\triangle ABC$. Because $\overline{EF}$ is parallel to $\overline{BC}$, the Parallel Theorem says that

$$\frac{AE}{AB} = \frac{EF}{BC}.$$

This tells us that the corresponding sides of the two rectangles are proportional. Thus, the rectangles are scaled copies.

Remember that, to show two rectangles are scaled copies, you just need to show that the ratios of their sides are equal.

b. Label the two rectangles $ABCD$ and $LMNP$, and call O the intersection point of the four dashed line segments:



Since $\overline{ML}$ is parallel to $\overline{BA}$, apply the Parallel Theorem to $\triangle OBA$ and $\triangle OML$ to get

$$\frac{OM}{OB} = \frac{ML}{BA}.$$

Since $\overline{MN}$ is parallel to $\overline{BC}$, apply the same theorem to $\triangle OBC$ and $\triangle OMN$ to get

$$\frac{OM}{OB} = \frac{MN}{BC}.$$

$\frac{OM}{OB}$ appears in both equations, implying that

$$\frac{ML}{BA} = \frac{MN}{BC}.$$

Thus, the rectangles are scaled copies.

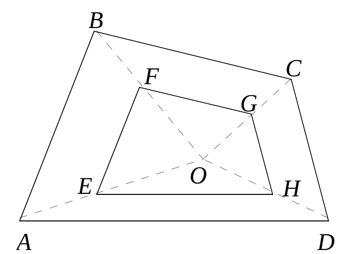
SIDE-SPLITTING AND PARALLEL THEOREMS
(continued)

Problem 14 *(Student page 82)* Since $\overline{DE} \parallel \overline{BC}$, it follows that corresponding angles of the two triangles are congruent. Moreover, the Parallel Theorem says that

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{ED}{CB}.$$

Thus all corresponding sides have the same ratio. It follows that the two triangles are scaled copies.

Problem 15 *(Student page 82)* Because the corresponding sides of the two polygons are parallel, it follows that corresponding angles are congruent. To illustrate this, let's work through the proof that $m\angle ABC = m\angle EFG$.



Because $\overline{BC} \parallel \overline{FG}$, you know that

$$m\angle CBO = m\angle GFO,$$

and because $\overline{AB} \parallel \overline{EF}$, you know that

$$m\angle ABO = m\angle EFO.$$

Adding these two equations shows that

$$m\angle CBO + m\angle ABO = m\angle GFO + m\angle EFO,$$

or, in other words,

$$m\angle ABC = m\angle EFG.$$

Similarly, you can show that the other three pairs of corresponding angles have equal measure.

Now apply the parallel theorem to the four pairs of nested triangles in the figure:

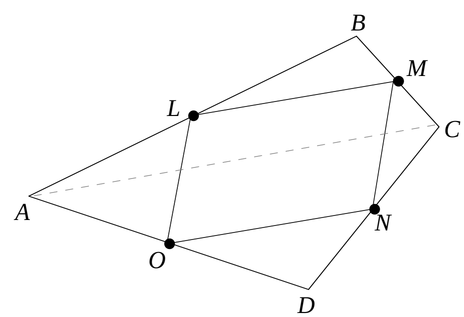
$$\frac{EF}{AB} = \frac{OF}{OB} = \frac{OE}{OA}$$
$$\frac{FG}{BC} = \frac{OF}{OB} = \frac{OG}{OC}$$
$$\frac{GH}{CD} = \frac{OG}{OC} = \frac{OH}{OD}$$
$$\frac{HE}{DA} = \frac{OH}{OD} = \frac{OE}{OA}$$

By looking for the ratios that appear more than once, you can deduce that all four of the following ratios are equal:

$$\frac{EF}{AB} = \frac{FG}{BC} = \frac{GH}{CD} = \frac{HE}{DA},$$

which proves that the two polygons are scaled copies.

Problems 16–17 *(Student page 83)* The inner quadrilateral obtained by joining the midpoints is a parallelogram. To prove this, add the dashed segment $\overline{AC}$ shown below:



Look at $\triangle ABC$. Points L and M are the midpoints of sides $\overline{AB}$ and $\overline{CB}$, respectively. Thus, by the Side-Splitting Theorem,

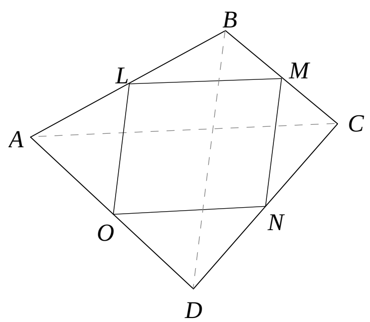
$$\overline{LM} \parallel \overline{AC}.$$

Applying the same reasoning to $\triangle ADC$ gives

$$\overline{ON} \parallel \overline{AC}.$$

SIDE-SPLITTING AND PARALLEL THEOREMS
(continued)

Therefore, $\overline{LM}$ and $\overline{ON}$ are parallel to each other, since they are both parallel to $\overline{AC}$. Now, apply the Side-Splitting Theorem to $\triangle ABD$ and $\triangle CBD$ to show that sides $\overline{LO}$ and $\overline{MN}$ are also parallel (as both are parallel to $\overline{BD}$). It follows that $LMNO$ is a parallelogram.



Problem 18 *(Student page 84)* Let $LMNO$ be the parallelogram obtained by joining consecutive midpoints of $ABCD$. Suppose that diagonal $\overline{DB}$ is 12 inches long and diagonal $\overline{AC}$ measures 8 inches.

Because $\overline{OL} \parallel \overline{DB}$ (as in the last problem), the Parallel Theorem says that

$$\frac{AO}{AD} = \frac{OL}{DB}.$$

But since O is a midpoint,

$$\frac{AO}{AD} = \frac{1}{2},$$

so

$$\frac{OL}{DB} = \frac{1}{2}.$$

This means that

$$OL = \frac{DB}{2} = \frac{12}{2} = 6.$$

Since opposite sides of a parallelogram are the same length, it follows that $NM = 6$, also.

Applying the same reasoning to the parallel sides $\overline{ON}$ and $\overline{AC}$ gives $ON = 4$ $(= LM)$. Thus, the perimeter of $LMNO$ is equal to $6 + 6 + 4 + 4 = 20$ inches.

Problem 19 *(Student page 84)*

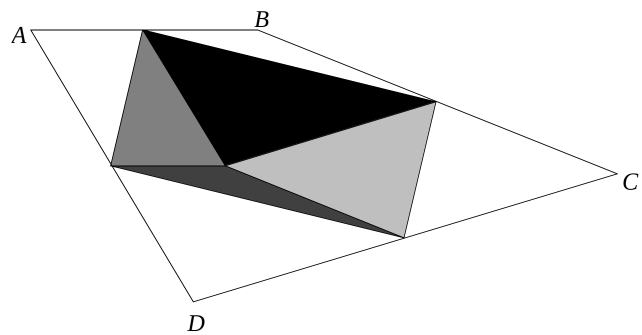
a. If the diagonals of the outer quadrilateral are congruent, the midpoint quadrilateral is a rhombus.

b. If the diagonals are perpendicular to each other, the midpoint quadrilateral is a rectangle.

c. Putting these two together, if the diagonals are both congruent and perpendicular, the midpoint quadrilateral is a "rectangular rhombus." In other words, it is a square.

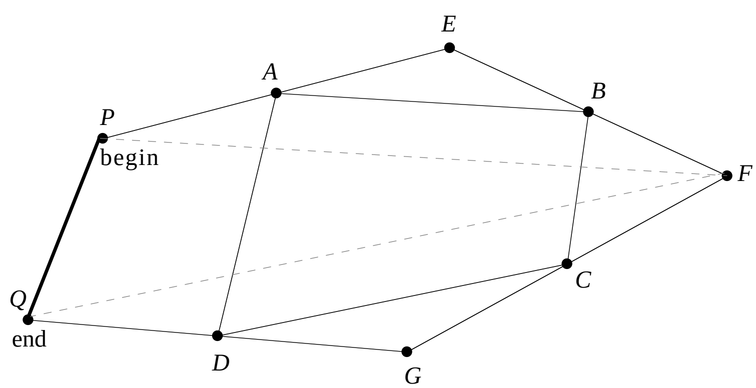
Can you prove why this works?

Problem 20 *(Student page 84)* The four outer triangles can be arranged inside the midpoint parallelogram to exactly cover it:



Thus, the area of the parallelogram is exactly half the area of the original quadrilateral *ABCD*.

Problem 21 *(Student page 84)* Let P be the starting point, Q the ending point, and E, F, and G the points created during the construction. The picture below also contains the dashed segments $\overline{PF}$ and $\overline{QF}$.



a. As you drag point P, notice that the length of $\overline{PQ}$ remains constant, as does its slope. But why?

Before reading the proof, drag the "begin" point (point P) around the screen. Notice which points move when point P moves and which stay in place.

Points A, B, C, and D of the quadrilateral stay fixed while E, F, G, and Q wander around the screen.

Now for the proof:

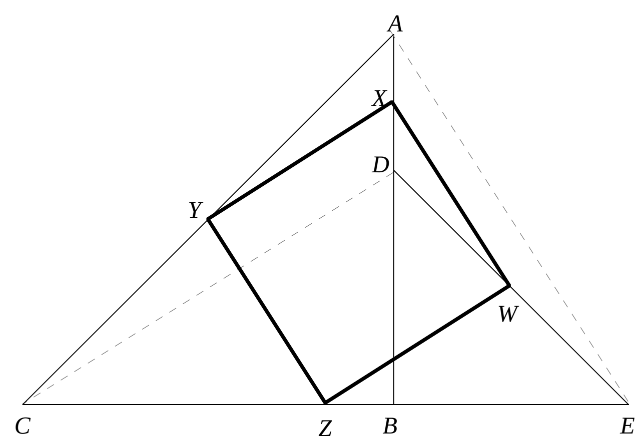
Look at $\triangle PEF$. Since points A and B are midpoints of sides $\overline{PE}$ and $\overline{FE}$ respectively, $\overline{PF}$ is twice the length of $\overline{AB}$ and parallel to it. Notice that moving point P does not affect $\overline{AB}$ at all—thus, the length and slope of $\overline{PF}$ also remain unchanged as point P moves.

Applying the same reasoning to $\triangle QGF$ shows us that the length and slope of $\overline{QF}$ also remain unchanged as point P moves.

Let's apply this information to $\triangle PFQ$. How do the shape and size of $\triangle PFQ$ depend on the location of point P? Imagine that for each different location of point P, you draw a picture of $\triangle PFQ$. From above, we know that the lengths of $\overline{PF}$ and $\overline{QF}$ will be the same in all of these triangles. We also know from above that $\angle PFQ$ will always be the same (why?). So by SAS, each of these versions of $\triangle PFQ$ will be congruent. Thus, the length of $\overline{PQ}$ remains the same regardless of point P's location. Furthermore, since the slopes of $\overline{PF}$ and $\overline{QF}$ do not change, neither will the slope of $\overline{PQ}$.

b. "Begin" and "end" are exactly the same point when $ABCD$ is a parallelogram.

Problem 22 *(Student page 85)* Let W, X, Y, and Z be the respective midpoints of sides $\overline{ED}$, $\overline{DA}$, $\overline{AC}$, and $\overline{CE}$. The goal is to show that $WXYZ$ is a square.



- Add dashed segments $\overline{AE}$ and $\overline{CD}$. First, look at $\triangle DAE$. Since X and W are midpoints of sides $\overline{DA}$ and $\overline{DE}$, the Side-Splitting Theorem says that $\overline{XW} \parallel \overline{AE}$.

SIDE-SPLITTING AND PARALLEL THEOREMS
(continued)

Then, by the Parallel Theorem, $XW = \frac{1}{2}AE$.

- Applying the same reasoning to $\triangle ACE$ gives you $\overline{YZ} \parallel \overline{AE}$ and $YZ = \frac{1}{2}AE$.
- Turning your attention to $\triangle CAD$, you discover that $\overline{XY} \parallel \overline{CD}$ and $XY = \frac{1}{2}CD$.
- Finally, from $\triangle CED$, $\overline{ZW} \parallel \overline{CD}$ and $ZW = \frac{1}{2}CD$.

Combining all the information above we see that

- $\overline{XW}$ and $\overline{YZ}$ are both half the length of $\overline{AE}$ and parallel to it.
- $\overline{XY}$ and $\overline{ZW}$ are both half the length of $\overline{CD}$ and parallel to it.

Now imagine rotating $\triangle CBD$ by 90° clockwise about B. D goes to E, since $DB = BE$ and $\angle DBE$ is a right angle. Further, C goes to A, since $CB = AB$ and $\angle CBA$ is a right angle. This implies that $\triangle CDB$ will lie directly on top of $\triangle AEB$ after it is rotated. Thus

$$CD = AE$$

and

$$\overline{CD} \perp \overline{AE}.$$

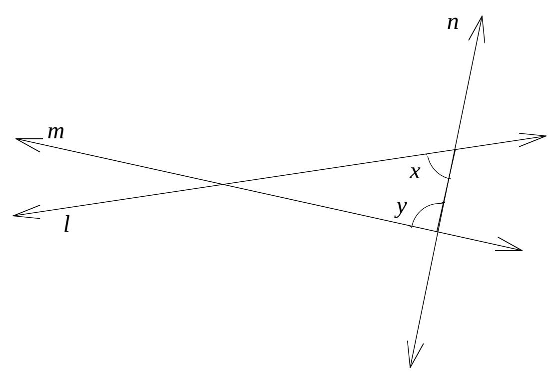
Combining the fact that $CD = AE$ with (1) and (2) from above, you see that the four sides of $WXYZ$ are congruent.

Combining the fact that $\overline{CD} \perp \overline{AE}$ with (1) and (2) from above, you see that $WXYZ$ has four right angles.

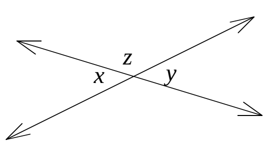
Thus, putting everything together, you can conclude that $WXYZ$ is a square.

Historical Perspective: Parallel Lines

Problem 1 *(Student page 88)* The trickiest part here is Euclid's fifth axiom. It basically says that, if two lines are not parallel, then they must intersect. The picture looks like this:



Lines l and m are cut by line n, and x and y are the measures of the indicated angles. The axiom says that since the sum of the measures of angles x and y is less than 180°, lines l and m must eventually intersect to the left of line n.



Problem 2 *(Student page 88)* In the picture, x and y are the measures of a pair of vertical angles. Since a straight line measures 180°,

$$x + z = 180^\circ$$

and

$$y + z = 180^\circ.$$

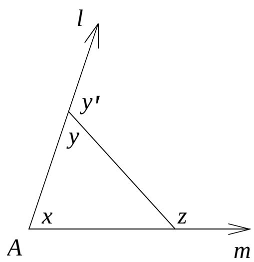
Therefore,

$$x + z = y + z$$

and

$$x = y.$$

Thus, the vertical angles have the same measure; that is, vertical angles are congruent.



Problem 3 *(Student page 88)* Let's see what happens if we assume $z < y$. Notice that

$$y + y' = 180^\circ,$$

since the two angles with these measures form a straight line.

The fact that $z < y$ implies

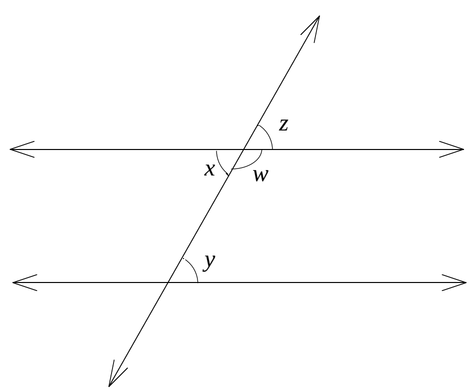
$$z + y' < 180^\circ.$$

Now apply Euclid's fifth axiom. The axiom says that the two lines l and m must intersect somewhere to the *right* of y' and z. But these two lines already intersect at A, and two distinct lines cannot intersect twice. Thus, our starting assumption that $z < y$ was false. Therefore, it must be that $z \geq y$. In a similar way, you can show that $z \geq x$.

This type of proof is called a *proof by contradiction*—you assume the opposite of what you want to prove and then show that this leads to a false conclusion.

Problem 4 *(Student page 89)*

a. We're asked to show that $x = y$. Let's start by assuming that the angles with measures x and y are *not* congruent, and see if this leads to any contradictions.



Assume that $y > x$. We know that

$$x + w = 180^\circ.$$

Thus

$$y + w > 180^\circ.$$

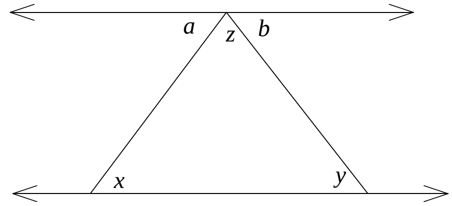
But then, by Euclid's fifth axiom, the two horizontal lines must intersect. This is a contradiction since we are given that the lines are parallel.

You will get the same contradiction if you assume that $y < x$. Thus, alternate interior angles x and y must be congruent.

b. x and z are the measures of vertical angles, so they are equal. Since you just showed that $x = y$, it must also be true that $z = y$.

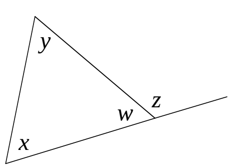
Problem 5 *(Student page 89)* In the picture below,

$$a + z + b = 180^\circ.$$



Because the two horizontal lines are parallel, the alternate interior angles are congruent, so $a = x$ and $b = y$. Making these substitutions into the equation above, you see that

$$x + z + y = 180^\circ.$$



Problem 6 *(Student page 90)* Consider a triangle with angle measures w, x, and y, with z measure of the exterior angle adjacent to the angle with measure w. From Problem 5, you know that

$$w + x + y = 180^\circ.$$

Since w and z form a straight line,

$$w + z = 180^\circ.$$

Thus

$$w + z = w + x + y,$$

giving

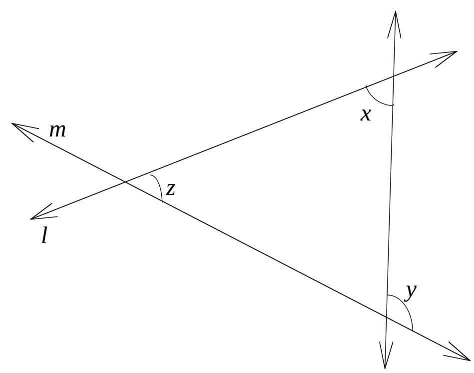
$$z = x + y,$$

which is what you wanted to show.

HISTORICAL PERSPECTIVE: PARALLEL LINES
(continued)

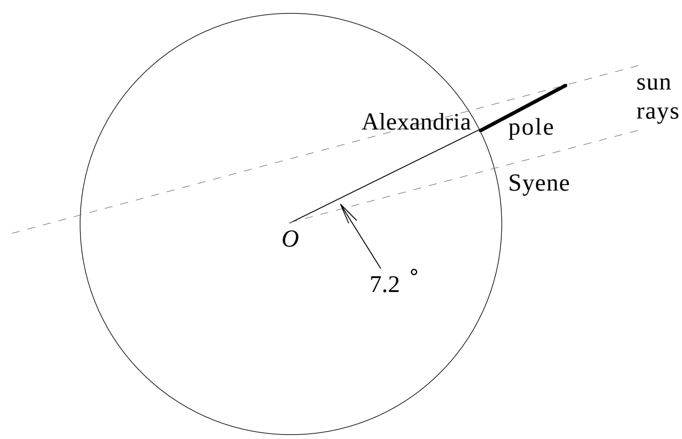
Because x and y are both positive numbers, z must be greater than either x or y, since z is equal to their sum.

Problem 7 *(Student page 90)* Suppose lines l and m are cut by a transversal, and congruent alternate interior angles with measures x and y are formed. You need to show that lines l and m are parallel. To prove this, let's do a proof by contradiction. Assume the lines are *not* parallel; then they must intersect somewhere. Let z be the measure of the angle formed by their intersection. As you can see, the intersection of the lines, together with the transversal, form a triangle:



By Problem 6, the exterior angle with measure y of this triangle must be larger than the angle with measure x. This is a contradiction, since we are given that these angles are congruent. Thus, our assumption that l and m intersect is false, and the two lines l and m are, in fact, parallel.

Problem 8 *(Student page 91)* The angle at O measures 7.2°. Since the sun's rays are parallel, the angle at O and the angle measuring 7.2° are equal alternate interior angles.



An interesting fact: the meter was originally defined as the length necessary to guarantee that the distance from the North Pole to the equator would be 10,000,000 meters.

Problems 9–10 *(Student page 91)* From the previous problem, you saw that the distance between Alexandria and Syene—500 miles—corresponds to an angle of 7.2° at angle O. Since a full circle is 360° and $\frac{360}{7.2} = 50$, the circumference of the Earth is $500(50) = 25{,}000$ miles by Eratosthenes's method.

The Earth is actually shaped like a pear, not a sphere. Its circumference at the equator is 24,901.5 miles and 24,855 at the poles. Eratosthenes was pretty clever!

DEFINING SIMILARITY

Problems 1–2 *(Student page 93)* Yes, Girt is a dilated copy of Trig. Copy the picture and draw rays through pairs of corresponding features (eyes, nose, ears, and so on). You will see that all these rays intersect at a point. This point is the center of dilation, and is located to the right of Girt. The pictures are similar by the dilation definition.

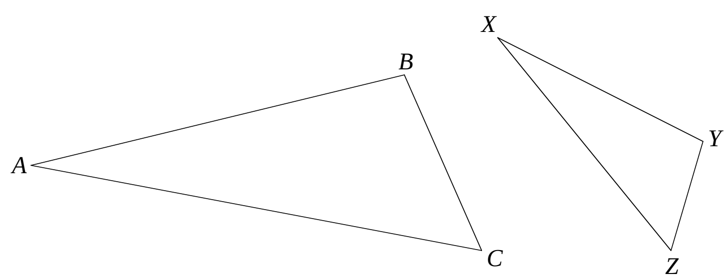
Problems 3–5 *(Student page 93)* In this picture, Girt has been reflected so that she faces the opposite direction. Now the pictures of Trig and Girt cannot be considered dilations of each other, since dilation preserves orientation. However, the pictures of Trig and Girt are still scaled copies of each other, so they are similar.

Problem 6 *(Student page 94)* Congruent figures are similar (with a scale factor of 1), but similar figures are not necessarily congruent.

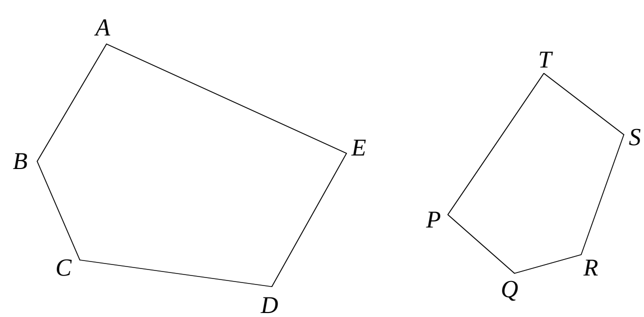
Problem 7 *(Student page 94)* If a figure A and a figure B are both scaled copies of a figure C (they are both similar to C), then certainly figures A and B are scaled copies of each other. Therefore, A and B are similar.

Problem 8 *(Student page 96)* The vertices should be labeled as follows:

a.



b.



DEFINING SIMILARITY (continued)

Problem 9 *(Student page 97)* Because $\triangle NEW \sim \triangle OLD$, it follows that

$$\begin{aligned} \angle N &\cong \angle O \\ \angle E &\cong \angle L \\ \text{and} \qquad \angle W &\cong \angle D. \end{aligned}$$

Since $m\angle N = 19^\circ$, you know that $m\angle O = 19^\circ$, and since $m\angle L = 67^\circ$, you know that $m\angle E = 67^\circ$. Adding the angle measures in $\triangle NEW$ tells you that

$$\begin{aligned} m\angle N + m\angle E + m\angle W &= 180^\circ \\ 19^\circ + 67^\circ + m\angle W &= 180^\circ \\ m\angle W &= 94^\circ. \end{aligned}$$

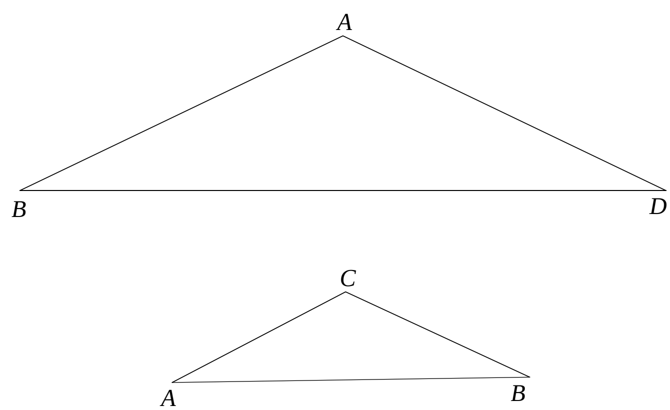
This also tells you that $m\angle D = 94^\circ$, which completes the problem.

Problem 10 *(Student page 97)* Because $\triangle ABC \sim \triangle DEF$, you know that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

because corresponding sides of similar triangles are proportional. These equalities show that statements a and b are true and statement c is false. Statement d is also true; it follows from cross multiplication of the first and third ratios above.

Problem 11 *(Student page 97)* Here is a picture of $\triangle BAD$ and $\triangle ACB$, both oriented in the same position:



DEFINING SIMILARITY *(continued)*

Why is $\triangle BAD$ also isosceles?

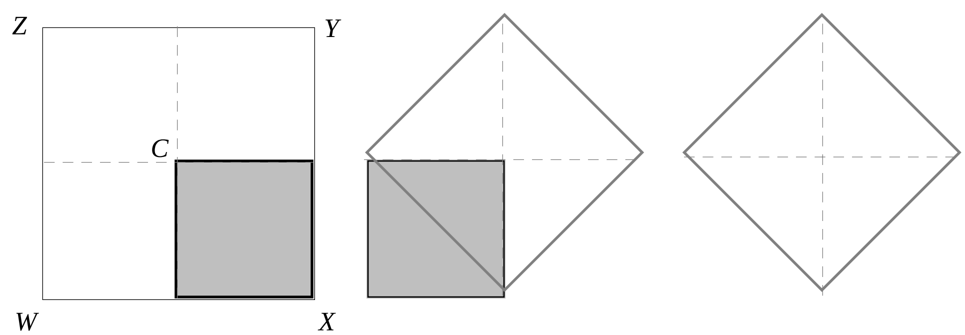
Since the triangles are similar, $\angle ABD \cong \angle CAB$. If you look back at the picture in the Student Module, you'll see that $\angle ABD$ is the same angle as $\angle ABC$. Thus, $\angle ABC \cong \angle CAB$, proving that $\overline{AC} \cong \overline{BC}$. Therefore, $\triangle ACB$ is isosceles.

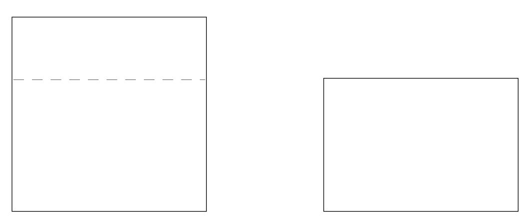
Problem 12 *(Student page 97)* The "Ways to Think About It" following this problem is really the solution to the problem.

Problem 13 *(Student page 99)* One way to solve this problem is to try to fold the required sidelengths of $\frac{3}{\sqrt{2}}$ and $\frac{4}{\sqrt{2}}$. The following solution is a little simpler.

Fold the square as follows:

Fold each vertex of $WXYZ$ into the center C of the square to create a square with half the area of the original square. Then, fold one quarter of this square down to create a rectangle that is similar to and has one half the area of the original 3×4 rectangle.





SIMILAR TRIANGLES

Problem 1 *(Student page 100)* AAA, SAS, and SSS are valid similarity tests. The SA test is not valid. It doesn't make sense to say that two triangles have just one pair of corresponding sidelengths proportional—what are they proportional to? You need at least two pairs of measurements to discuss proportionality, since a proportion is an equation stating that two (or more) ratios are equal.

Problems 2–3 *(Student page 102)* The AAA test can actually be replaced by an AA test. Once you know two angles of a triangle, the measurement of the third angle is completely determined since the angle sum is 180°. Thus, the theorem can be rewritten as follows:

> If two triangles have two pairs of corresponding angle measures congruent, then the triangles are similar.

Problem 4 *(Student page 102)* An ASA test doesn't make sense. As noted above in Problem 1, you can't say that just one pair of sidelengths are proportional.

Problem 5 *(Student page 102)* An AAAA similarity test does not work for quadrilaterals. For example, any two rectangles share the same angle measurements, but are not necessarily similar.

Problem 6 *(Student page 102)* Recall that when parallel lines are cut by a transversal, corresponding angles are congruent. Since $\overline{DG} \parallel \overline{BC}$, it follows that

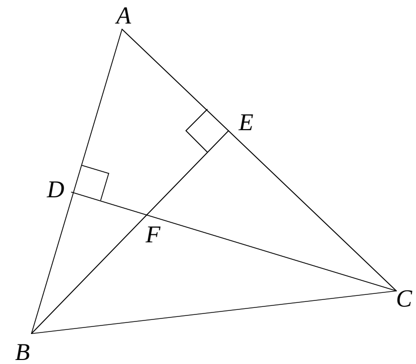
$$\angle HGF \cong \angle BCA,$$

and because $\overline{EF} \parallel \overline{BA}$, you know that

$$\angle HFG \cong \angle BAC.$$

This gives two pairs of congruent corresponding angles. By the AA similarity test, this is enough to show that $\triangle ABC \sim \triangle FHG$.

Problem 7 *(Student page 103)* To start, you know that $\overline{CD}$ and $\overline{AB}$ are perpendicular, as are $\overline{BE}$ and $\overline{AC}$.



You can show that the following pairs of triangles are similar:

- $\triangle DFB \sim \triangle EFC$: $\angle FDB$ and $\angle FEC$ are both right angles, and $\angle DFB \cong \angle EFC$ because they are vertical angles. By the AA test, the triangles are similar.
- $\triangle AEB \sim \triangle ADC$: Angles $\angle AEB$ and $\angle ADC$ are both right angles, and $\angle A \cong \angle A$. By AA, the triangles are similar.

Problem 8 *(Student page 103)* Since both triangles share $\angle A$, and $\angle ADE$ and $\angle ACB$ are congruent, the triangles are similar by AA.

Problem 9 *(Student page 104)* Notice that by the AA test,

$$\triangle ACB \sim \triangle AED \sim \triangle AGF,$$

since they all share $\angle A$ and each contains a right angle.

This means that corresponding sides are proportional. In particular,

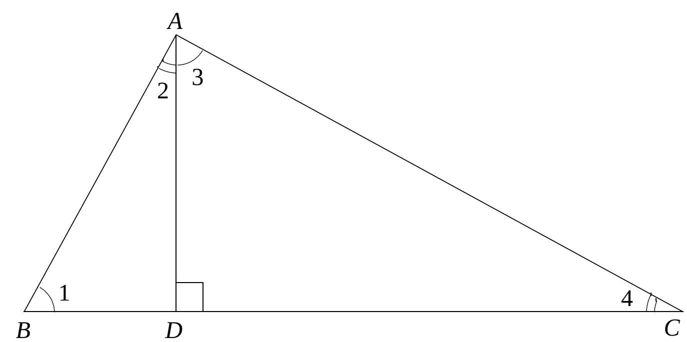
$$\frac{AB}{BC} = \frac{AD}{DE} = \frac{AF}{FG}.$$

Since $AB = 2$ and $BC = 1$, you know that

$$\frac{AB}{BC} = 2,$$

and so the ratios $\frac{AD}{DE}$ and $\frac{AF}{FG}$ are also equal to 2.

Problem 10 *(Student page 104)* Start with right triangle ABC, with the right angle at A; D will be the point at which the altitude from A intersects $\overline{BC}$. Label the acute angles 1, 2, 3, and 4 as shown in the picture:



One way to find pairs of congruent angles is to see which pairs have measures that add up to 90°. Notice that the sum of the measures of the two acute angles in a right triangle must be 90°. Therefore,

$$\begin{aligned} m\angle 1 + m\angle 4 &= 90^\circ \\ m\angle 1 + m\angle 2 &= 90^\circ \\ m\angle 3 + m\angle 4 &= 90^\circ. \end{aligned}$$

You can use these statements to show that

$$\angle 2 \cong \angle 4$$

and

$$\angle 1 \cong \angle 3.$$

Then, by AA similarity,

$$\begin{aligned} \triangle ABC &\sim \triangle DBA \\ \triangle ABC &\sim \triangle DAC \end{aligned}$$

and

$$\triangle DBA \sim \triangle DAC.$$

Thus, all three triangles are similar:

$$\triangle ABC \sim \triangle DBA \sim \triangle DAC.$$

(When you write these similarity statements, make sure that you write corresponding vertices in corresponding positions.)

Problem 11 *(Student page 104)* Suppose you have two triangles, $\triangle ABC$ and $\triangle DEF$, such that

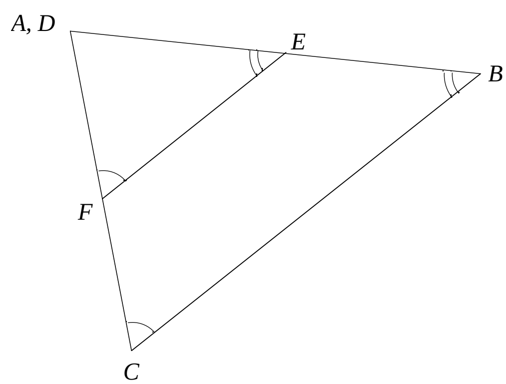
$$\angle A \cong \angle D$$

and

$$\frac{AC}{DF} = \frac{AB}{DE}.$$

These are the hypotheses of the SAS test; you now need to show that the triangles are similar.

The key is to arrange the triangles one inside the other so that the congruent angles at A and D line up:



Because of the equal ratios given above, the Side-Splitting Theorem implies that $\overline{FE}$ is parallel to $\overline{CB}$, so you know that

$$\angle DFE \cong \angle ACB$$

and

$$\angle DEF \cong \angle ABC.$$

Moreover, the parallel theorem tells you that

$$\frac{AC}{DF} = \frac{AB}{DE} = \frac{CB}{FE}.$$

Thus, corresponding angles of the two triangles are congruent and corresponding sides are proportional, so

$$\triangle ABC \sim \triangle DEF.$$

This proves the SAS similarity test.

Problem 12 *(Student page 105)* Both $\triangle ABC$ and $\triangle EDC$ share $\angle C$, and from the given measurements, you can check that

$$\frac{AC}{EC} = \frac{BC}{DC} = 2.$$

This is enough to apply the SAS similarity test (the congruent angles are the included ones between the two pairs of proportional sides) and conclude that

$$\triangle ABC \sim \triangle EDC.$$

It follows from the similarity that the third pair of side lengths must also be in proportion to the other two pairs, so

$$\frac{AB}{ED} = 2.$$

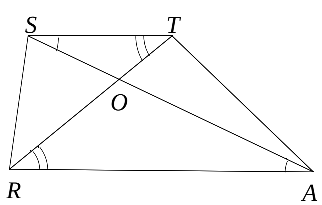
Since

$$AB = 4,$$

you know that

$$DE = 2.$$

Problem 13 *(Student page 105)*



a. Notice that

$$\angle TSO \cong \angle RAO$$

and

$$\angle STO \cong \angle ARO,$$

since both pairs are alternate interior angles of the parallel lines containing $\overline{ST}$ and $\overline{RA}$.

This shows that

$$\triangle ROA \sim \triangle TOS$$

by the AA test. (You could also have shown that the remaining two angles are congruent, as they are vertical angles.)

b. In similiar triangles, the ratios of corresponding sides are equal. Thus,

$$\frac{RO}{TO} = \frac{OA}{OS}.$$

c. You cannot apply the SAS similarity test here. The test says that two pairs of corresponding sides must be proportional and the included angles between these two sides must be congruent. We know from the previous page that

$$\frac{RO}{TO} = \frac{OA}{OS}.$$

But to show that $\triangle ROS \sim \triangle TOA$, we would need

$$\frac{RO}{TO} = \frac{OS}{OA}.$$

Problem 14 *(Student page 106)* In the proofs of the AAA and SAS tests, you had at least one pair of corresponding angles that you knew was congruent. This meant you could fit one triangle inside the other and the congruent angles would align perfectly. In the case of SSS, however, you have no congruent angles to use.

Problem 15 *(Student page 107)* Since F, G, and H are midpoints of $\triangle ABC$, the Side-Splitting Theorem tells us that the sides of $\triangle GHF$ are parallel to the sides of $\triangle ABC$. And, by the Parallel Theorem, each side of $\triangle GHF$ is half as long as the corresponding side of $\triangle ABC$. Thus

$$\frac{1}{2} = \frac{GH}{AB} = \frac{HF}{BC} = \frac{GF}{AC}.$$

Then, by the SSS test,

$$\triangle GHF \sim \triangle ABC.$$

Problem 16 *(Student page 108)* You can apply the SSS test to get

$$\triangle ABC \sim \triangle AFH \sim \triangle FBG \sim \triangle HGC.$$

Problem 17 *(Student page 108)* If a triangle is similar to a triangle with sides of 4, 5, and 8, its sides must have lengths $4x$, $5x$, and $8x$, respectively. Since we don't know which side is of length 3, there are three possibilities:

- If $4x = 3$, then its sides measure 3, $\frac{15}{4}$, and 6.
- If $5x = 3$, then its sides measure $\frac{12}{5}$, 3, and $\frac{24}{5}$.
- Finally, if $8x = 3$, its sides measure $\frac{3}{2}$, $\frac{15}{8}$, and 3.

Problem 18 *(Student page 108)* If a triangle is similar to a triangle with sides 2, 3, and 4 inches, its sides must have lengths $2x$, $3x$, and $4x$ inches, respectively. Since you are told that its perimeter is 6 inches, you can write

$$6 = 2x + 3x + 4x = 9x.$$

Thus $x = \frac{2}{3}$, and the sides of the triangle are $\frac{4}{3}$, 2, and $\frac{8}{3}$.

Problem 19 *(Student page 108)*

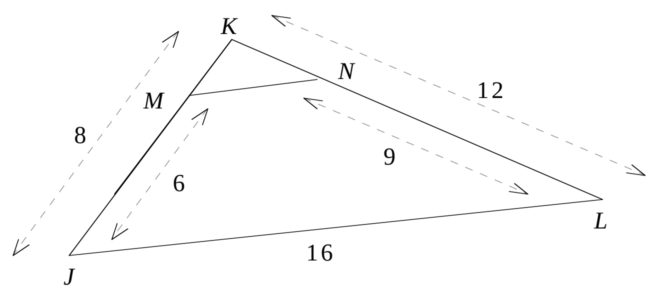
a. Points M and N split sides $\overline{JK}$ and $\overline{KL}$ proportionally, since

$$\frac{KM}{KJ} = \frac{2}{8} = \frac{1}{4}$$

and

$$\frac{KN}{KL} = \frac{3}{12} = \frac{1}{4}.$$

By the Side-Splitting Theorem, $\overline{MN}$ and $\overline{JL}$ are parallel.



b.

- Since $\overline{MN} \parallel \overline{JL}$, the following corresponding angles are congruent:

  $$\angle KMN \cong \angle KJL$$

  and

  $$\angle KNM \cong \angle KLJ.$$

  The AA test shows that $\triangle MKN \sim \triangle JKL$.

- From part a,

  $$\frac{KM}{KJ} = \frac{KN}{KL}.$$

  Since $\angle JKL \cong \angle MKN$, we know that $\triangle MKN \sim \triangle JKL$ by the SAS test.

- By the Parallel Theorem,

  $$\frac{KM}{KJ} = \frac{KN}{KL} = \frac{MN}{JL}.$$

  Thus, by the SSS test, $\triangle MKN \sim \triangle JKL$.

Problem 20 *(Student page 108)* To start off, label the sidelengths of $\triangle ABC$ as follows:

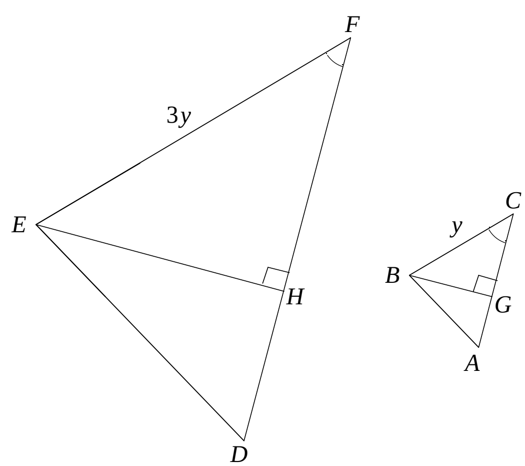
$$\begin{aligned} AB &= x \\ BC &= y \\ CA &= z. \end{aligned}$$

It follows that the sides of $\triangle DEF$ have the following lengths:

$$\begin{aligned} DE &= 3x \\ EF &= 3y \\ FD &= 3z. \end{aligned}$$

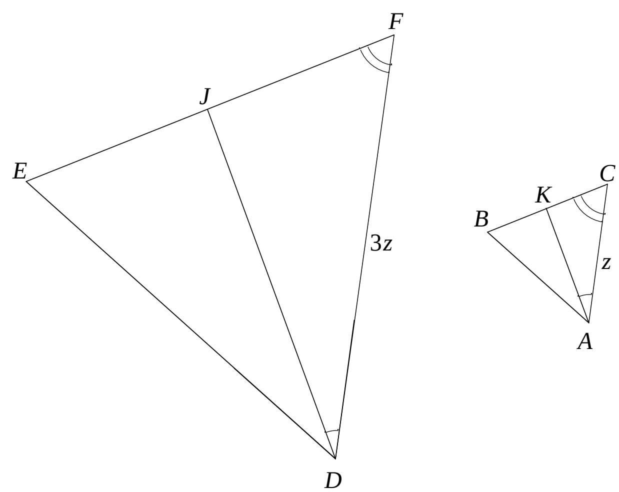
a. Let $\overline{BG}$ and $\overline{EH}$ be altitudes of $\triangle ABC$ and $\triangle DEF$. Since $\angle EFD \cong \angle BCA$, and $\angle H$ and $\angle G$ are both right angles, the AA test shows that $\triangle EFH \sim \triangle BCG$. Thus,

$$\frac{EH}{BG} = \frac{EF}{BC} = \frac{3y}{y} = 3.$$



b. Draw angle bisectors $\overline{DJ}$ and $\overline{AK}$. Because $\angle FDE \cong \angle CAB$, it follows that $\angle FDJ \cong \angle CAK$. Because $\angle EFD \cong \angle BCA$, the AA test shows that $\triangle JFD \sim \triangle KCA$. Thus,

$$\frac{DJ}{AK} = \frac{DF}{AC} = \frac{3z}{z} = 3.$$

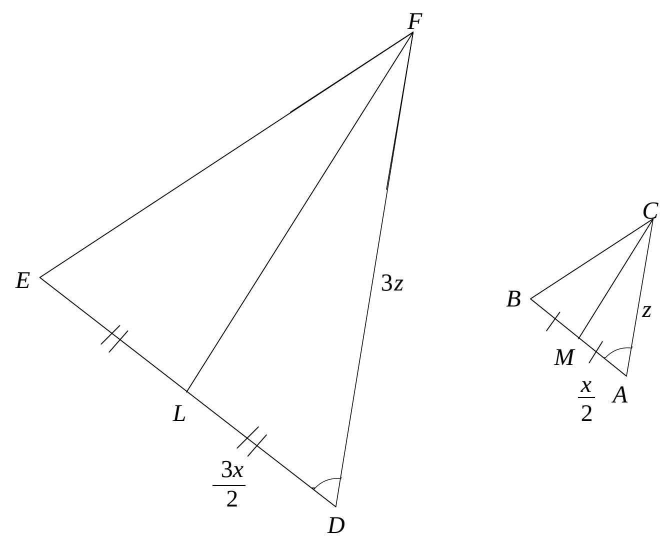


c. Construct medians $\overline{FL}$ and $\overline{CM}$. Then $LD = \frac{3x}{2}$ and $MA = \frac{x}{2}$, implying that

$$\frac{LD}{MA} = \frac{FD}{CA} = 3.$$

Combine this with $\angle EDF \cong \angle BAC$ to see that $\triangle FLD \sim \triangle CMA$ by SAS. Therefore,

$$\frac{FL}{CM} = \frac{FD}{CA} = \frac{3z}{z} = 3.$$



Problem 21 *(Student page 109)* Suppose that the altitudes in $\triangle ABC$ have lengths h_A, h_B, and h_C, where the subscript indicates the vertex from which the altitude is drawn, and suppose also that the altitudes in $\triangle DEF$ have lengths h_D, h_E, and h_F. The given information tells you that

$$\begin{aligned} h_A &= h_D \\ h_B &= h_E \\ \text{and} \qquad h_C &= h_F. \end{aligned}$$

Now, write the area of each triangle in three different ways:

$$\text{Area } (\triangle ABC) = \frac{1}{2}(BC)h_A = \frac{1}{2}(AC)h_B = \frac{1}{2}(AB)h_C$$

and

$$\text{Area } (\triangle DEF) = \frac{1}{2}(EF)h_D = \frac{1}{2}(DF)h_E = \frac{1}{2}(DE)h_F.$$

Substitute for the heights in $\triangle DEF$:

$$\text{Area } (\triangle DEF) = \frac{1}{2}(EF)h_A = \frac{1}{2}(DF)h_B = \frac{1}{2}(DE)h_C.$$

Look at the ratio of the area of $\triangle ABC$ to the area of $\triangle DEF$, writing this ratio three different ways:

$$\frac{\text{Area }(\triangle ABC)}{\text{Area }(\triangle DEF)} = \frac{BC}{EF} = \frac{AC}{DF} = \frac{AB}{DE}.$$

This shows that corresponding sidelengths of the two triangles are proportional, so

$$\triangle ABC \sim \triangle DEF$$

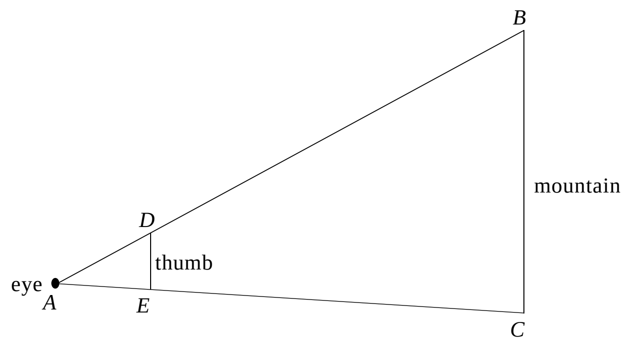
by the SSS test.

From Problem 20a, we know that the ratio of any two corresponding sides of these triangles is equal to the ratio of any two corresponding altitudes. Since $h_A = h_D$, we have $\frac{h_A}{h_D} = 1$. Thus $\frac{AB}{DE} = 1$ and $AB = DE$. If two similar triangles have a pair of corresponding sides congruent, then the triangles must, in fact, be congruent. Thus, $\triangle ABC$ is congruent to $\triangle DEF$.

USING SIMILARITY

Calculating Distances and Heights

Problems 1–2 *(Student page 111)* Assuming that the thumb is held parallel to the mountain, $\triangle ABC \sim \triangle ADE$ by the AA test.



Notice that each ratio in the proportion here compares inches to feet.

If $DE = 1$ inch, $AE = 14$ inches, and $BC = 6{,}600$ feet, then you can use the ratio

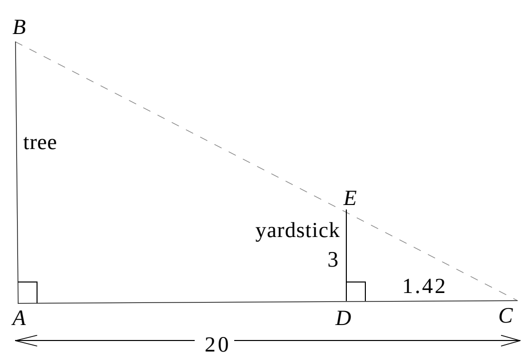
$$\frac{AE}{AC} = \frac{DE}{BC}$$

to calculate that

$$\frac{14}{AC} = \frac{1}{6{,}600},$$

so $AC = 92{,}400$ feet.

Problem 4 *(Student page 113)* In the picture below, $\overline{BC}$ represents a ray of light from the sun, $\overline{AB}$ represents the tree, and $\overline{DE}$ represents the yardstick. The shadows cast by the tree and the yardstick are represented by $\overline{AC}$ and $\overline{DC}$, respectively.



Because the tree and the yardstick are parallel,

$$\triangle ABC \sim \triangle DEC.$$

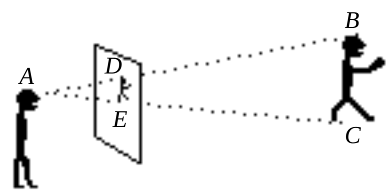
This similarity can be used to calculate the height of the tree. First convert the length of the yardstick's shadow into feet, so that all measurements will be in the same units.

You know that $AC = 20$ feet, $DC = 1.42$ feet, and $DE = 3$ feet. Therefore,

$$\frac{AB}{3} = \frac{20}{1.42},$$

so that AB (the height of the tree) is about 42.25 feet.

Problems 6–7 *(Student page 114)* Labels for important points have been added to the picture in the Student Module.



$\triangle ABC \sim \triangle ADE$, so you know that

$$\frac{DE}{BC} = \frac{AE}{AC}.$$

Therefore, since $BC = 5$ feet, $AC = 30$ feet, and $AE = 2$ feet,

$$\frac{DE}{5} = \frac{2}{30},$$

and $DE = \frac{1}{3}$ foot. The image will be about $\frac{1}{3}$ foot (or 4 inches) tall.

Problem 8 *(Student page 114)* To determine how tall the image of the planet would be, you need to know the actual height (diameter) of the planet, the distance between the planet and Earth, and the distance between you and the window.

Problem 9 *(Student page 114)* There are 31,536,000 seconds in a year (assuming the year has 365 days). Since light travels 186,000 miles per second, it travels

5.87×10^{12} miles per year. This means that the nearest star, four light-years away, is actually

$$4 \times 5.87 \times 10^{12} = 2.35 \times 10^{13}$$

miles away from Earth!

Problem 10 *(Student page 114)* You know that the distance from you to the planet is 2.35×10^{13} miles, the distance from you to the window is 2 feet, and the diameter of the planet is 7900 miles, since its diameter is the same as the Earth's. Using ratios as in the previous problems, you can calculate that the height of the planet as seen on the window will be 6.73×10^{-10} feet tall, or 8.08×10^{-9} inches tall!

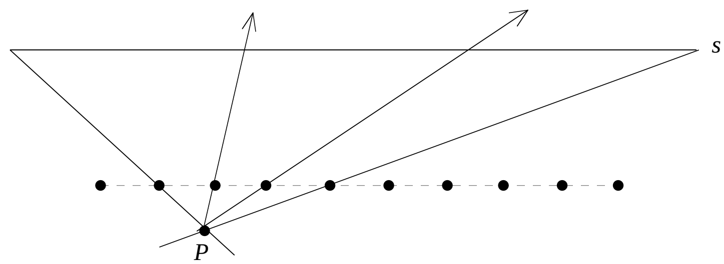
Problem 11 *(Student page 114)* An "o" on this page is approximately one half of one tenth of an inch wide, or one twentieth of an inch wide. What do you have to magnify 8.08×10^{-9} inches by to get $\frac{1}{20}$ of an inch? Solve the equation

$$(8.08 \times 10^{-9})(a) = \frac{1}{20}.$$

You will find that a is approximately 6,200,000. You would have to magnify the image of the planet by this much in order for it to be as big as an "o" on this page. No wonder we can't see these planets with our bare eyes!

Segment Splitters

Problems 12–13 *(Student page 116)* Suppose that you want to split segment s into three congruent pieces. Choose any *four* consecutive points on the dashed line, and then draw two lines: one connecting the left endpoint of s with the leftmost of the four points, and one connecting the right endpoint of s with the rightmost of the four points. Let P be the point where these lines meet:



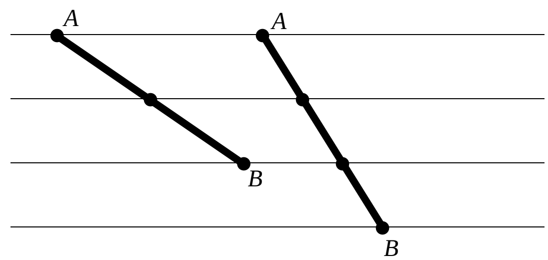
Now draw rays through P and each of the two middle points, extending both rays far enough to intersect s. The two points of intersection divide s into three congruent segments.

To divide s into 5 or 7 congruent pieces, repeat this process, but begin with 6 and 8 consecutive points, respectively, along the dashed line.

Problem 14 *(Student page 116)* You are really performing a projection here. When you divide s into three pieces, you begin by choosing four consecutive points on the dashed line. The dashed segment connecting the rightmost and leftmost points of the four points is being dilated onto segment s, with P as the center of dilation. Since the points are equally spaced on the dashed segment, their images under the dilation will be equally spaced on s, thereby dividing s into congruent pieces.

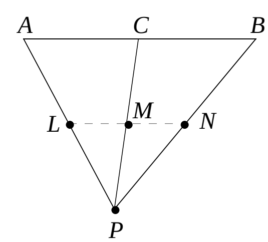
Problems 15–18 *(Student pages 116–117)* Draw a segment $\overline{AB}$ on the transparency. You can place the transparency over the notebook paper and see through it to the evenly-spaced parallel lines on the paper. To divide $\overline{AB}$ in half, position the transparency so that point A lies on a line, and point B lies on the line that is two lines away. The line in the middle will intersect $\overline{AB}$ through its midpoint, dividing it in half.

The figure below shows how to position $\overline{AB}$ to divide it into both two and three congruent pieces.



The largest number of divisions you can make depends on the length of your segment and the width of the spaces between the lines on your notebook paper. In general, you'll get the most divisions when $\overline{AB}$ is placed perpendicular to the lines on the paper.

Problem 19 *(Student page 117)* To prove that the projection method splits $\overline{AB}$ in half, let L, M, and N be the three points on the dashed segment, and let C be the intersection point of $\overline{AB}$ with the line through P and M:



By construction, the dashed segment is parallel to $\overline{AB}$. Apply the Parallel Theorem to $\triangle PAC$ and $\triangle PAB$ to obtain

$$\frac{PL}{PA} = \frac{LM}{AC}$$
$$\frac{PL}{PA} = \frac{LN}{AB}.$$

These two equations together imply that

$$\frac{LM}{AC} = \frac{LN}{AB},$$

or equivalently,

$$\frac{LM}{LN} = \frac{AC}{AB}.$$

Remember, though, that

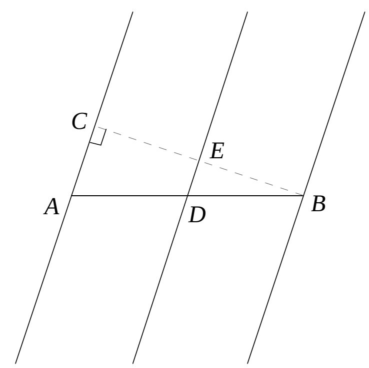
$$\frac{LM}{LN} = \frac{1}{2}$$

since the points on the dashed line are evenly spaced. Thus,

$$\frac{1}{2} = \frac{AC}{AB},$$

so C is the midpoint of $\overline{AB}$.

Now, let's show that the parallels method also works. Draw a perpendicular from B to the line containing A, and let C be the point where the perpendicular intersects this line. Let D and E be points on the middle line, as shown:



Because all three lines are parallel, you can apply the Parallel Theorem to $\triangle ABC$ to get

$$\frac{BE}{BC} = \frac{BD}{BA}.$$

Notice that

$$\frac{BE}{BC} = \frac{1}{2}$$

because all three parallel lines are evenly spaced. It follows that

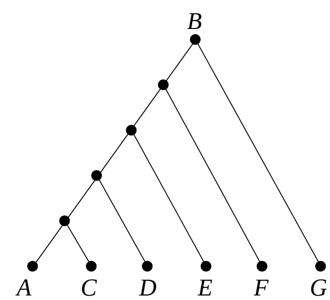
$$\frac{BD}{BA} = \frac{1}{2},$$

implying that point D is the midpoint of $\overline{AB}$.

Problem 20 *(Student page 117)* To show that these methods work for a division of a segment into any number of congruent parts, you use the same basic arguments as above, applying the Parallel Theorem several times.

Problems 21–22 *(Student page 118)* Let G be the final translated point. Connect G to B, and using the appropriate geometry software commands, construct lines

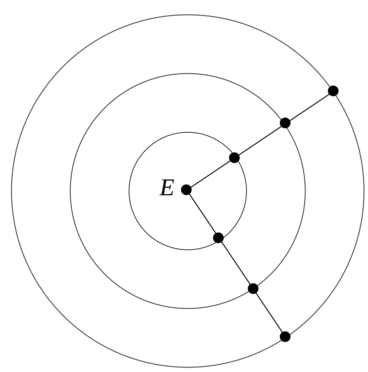
through C, D, E and F that are parallel to $\overline{GB}$. These lines will intersect $\overline{AB}$ at points that divide it into five congruent pieces.



This method is similar to the parallels method you tried earlier. Here, though, you're not given a piece of notebook paper with evenly-spaced, parallel lines. You use the computer to construct these lines instead.

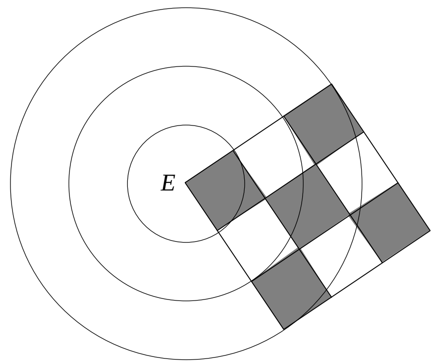
Problem 23 *(Student page 118)* Here's one way to construct a checkerboard. We'll create a checkerboard with 9 squares, but you can make a different size if you like. Start by trisecting a segment using the method from Problem 21. Then construct a line perpendicular to the segment at one of its endpoints, E. Now construct three concentric circles, all centered at E, and all intersecting the original line segment at the points that divide it into three equal pieces. Notice that these circles automatically trisect the perpendicular line segment you just drew:

***Concentric* circles are circles that share a center.**



Using Similarity *(continued)*

Use the appropriate software commands to construct the remaining sides of the square and the lines parallel to these sides to form the checkerboard:



The beauty of this construction is that you can easily change the length of the original segment, and everything else adjusts automatically.

Now, if you vary the length of the original segment, the square will grow or shrink with it.

Problem 24 *(Student page 123)* Because the marks along arm A are evenly spaced, the parallel lines drawn with arm B will also be evenly spaced. You can show this by using the same type of proof as the parallels method in Problem 19.

Problem 25 *(Student page 123)* Start with one endpoint of the segment on arm B and the other endpoint on the seventh mark of arm A. Then slide arm A one mark at a time to draw evenly spaced, parallel lines with arm B. The points where these parallel lines intersect your segment divide it into seven congruent parts.

Problem 26 *(Student page 123)* Several of the segment-splitting methods are similar in that they use equally-spaced parallel lines as tools of division. In the parallels method, you used notebook paper to provide the lines. With the geometry software method, you constructed the parallel lines using the software. The segment splitter is a “low-tech” version of the geometry software method, and it also creates parallel lines.

A Constant-Area Rectangle

Problem 29 *(Student page 124)* No matter which vertex or side of your *constructed* rectangle you move, it remains a rectangle—you cannot "mess" it up so that it becomes something else!

Problem 30 *(Student page 124)* In general, as you drag a vertex, the area, as well as the dimensions of the rectangle, will change.

Problems 31–33 *(Student page 125)* Rectangles with dimensions of $36' \times 1'$, $18' \times 2'$, $12' \times 3'$, and $9' \times 4'$ all have an area of 36 square feet. If you have any other rectangle with the same area and if its dimensions are $l \times w$, then it must be true that

$$l \cdot w = 36.$$

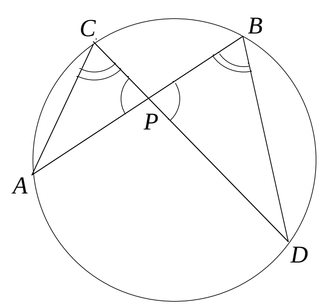
As long as you don't require your rectangle to have integer-length sides, you can generate an infinite number of rectangles with area of 36 square feet. Just let l be any positive number you want, and then w will equal $\frac{36}{l}$.

Problem 34 *(Student page 126)* For any given point P, the value $PA \cdot PB$ will be the same for *any* chord $\overline{AB}$ passing through P. This constant value depends on P's location, though. For a different interior point, such as P', the constant value of $P'A \cdot P'B$ may be different.

Problem 35 *(Student page 126)* You want to show that

$$PA \cdot PB = PC \cdot PD.$$

To do this, draw $\overline{AC}$ and $\overline{BD}$:



Angles $\angle CPA$ and $\angle BPD$ are congruent, as they are vertical angles. Moreover, angles $\angle ACD$ and $\angle ABD$ are congruent, as they are inscribed angles that intercept the same

arc of the circle. Therefore, by the AA similarity test, you know that

$$\triangle APC \sim \triangle DPB.$$

Since these triangles are similar,

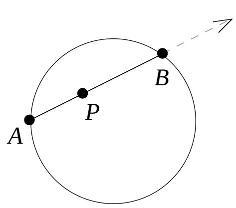
$$\frac{PC}{PB} = \frac{PA}{PD},$$

or equivalently,

$$PA \cdot PB = PC \cdot PD.$$

There was nothing special about the choice of points A, B, C, and D, so this relation will hold for any chords passing through point P.

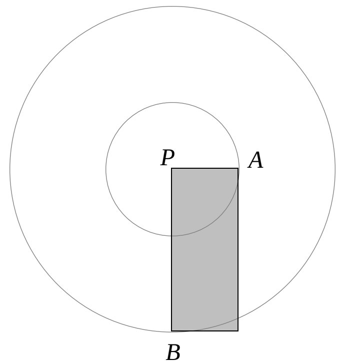
Problem 36 *(Student page 127)* The point P inside the circle is chosen to have its power equal to 12. To build a rectangle with an area of 12 square feet, you need two sidelengths whose product is 12. Take a stick of wood and lay it with one end on the circle's circumference, so that it passes through P. Cut the stick where it intersects the circle, and also cut it at the point that hits P. By cutting the stick, you're making it a chord of the circle that is divided into two pieces whose lengths multiply together to equal 12. Now cut a second stick into two pieces of the same size and use these four pieces to make a rectangle. To build another rectangle, take another stick and use it to form a different chord of the circle passing through P. You will then get two lengths whose product is 12, but they will be different from the lengths of the first rectangle (We should say they'll probably be different. See Problem 42).

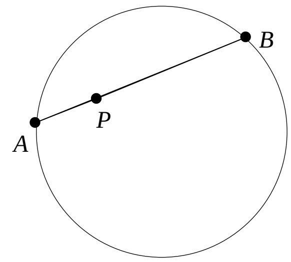


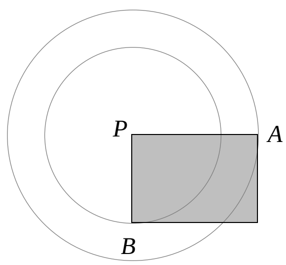
Problems 37–39 *(Student page 128)* To begin the construction, draw a circle along with a point P in its interior. Place a point A on the circle, and construct the ray through A and P. Let B be the second point of intersection of this ray with the circle. Construct segments $\overline{PA}$ and $\overline{PB}$ and then hide the ray. This guarantees that, as you move A around the circle, the chord will change, but it will always pass through P.

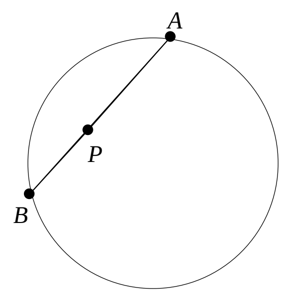
Now construct two concentric circles somewhere else on the screen, one with radius $\overline{PA}$ and one with radius $\overline{PB}$ (see the following picture). Construct radii at right angles to each other to form the length and width of a rectangle. Then construct the remaining two sides of the rectangle. This rectangle will have area $PA \cdot PB$. As you drag A around the original circle, the radii adjust themselves accordingly, causing the rectangle to grow and shrink. The area of the rectangle stays constant, however, since the power of P remains constant as A moves. Following are some pictures showing different locations of the chord $\overline{AB}$, and the corresponding rectangles. All have different dimensions, but share the same area.

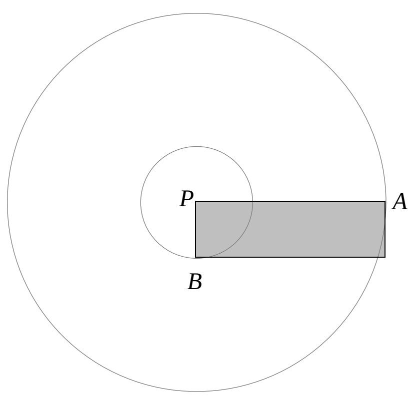
USING SIMILARITY (continued)

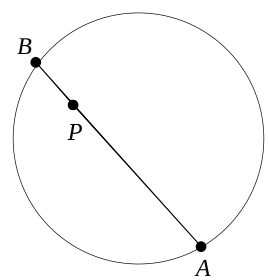






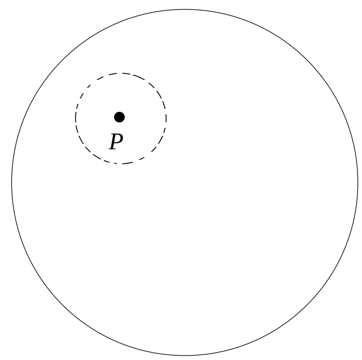






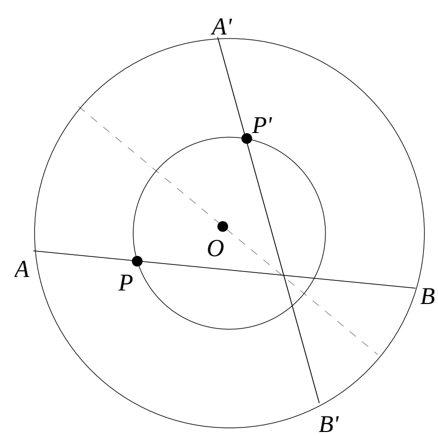
Problem 41 *(Student page 128)* Your construction is limited to a specific range of rectangles with the same area. For any particular location of P within the circle, the values of PA and PB cannot become arbitrarily small or large.

Try this: Look at the circle on your computer screen that contains P and the chord $\overline{AB}$. Construct *another* circle with P as its center (shown as a dashed circle in the picture below). The radius of the circle does not matter, but it should be adjustable. By experimenting with different radii of this circle, you can find the location of point A for which PA is the smallest. Can you figure out how?



The radius of the dashed circle is adjustable.

Problem 42 *(Student page 128)* Suppose you take a point P inside a circle with center O, along with a chord $\overline{AB}$ passing through P. Draw any diameter of the circle (the dashed line in the picture below), and reflect $\overline{AB}$ and P about this diameter, obtaining a new point P', and a new chord $\overline{A'B'}$.



Since reflection preserves distances between points, it follows that

$$PA = P'A'$$

and

$$PB = P'B'.$$

This implies that

$$PA \cdot PB = P'A' \cdot P'B',$$

so P and P' share the same power. Therefore, any such reflection about a diameter produces a point with the same power.

Notice that

$$OP = OP',$$

again because of the distance-preserving reflection. This means that P and P' lie on the same concentric circle inside the original circle.

So you now know that, given any point, a reflection about the diameter will produce another point with the same power and this point will lie on the circle centered at O with radius OP.

Thus, if you were to perform all possible reflections of point P across a diameter, you would obtain the circle centered at O with radius OP. All points on this circle have the same power as P.

Problem 43 *(Student page 129)* There is still a constant product when P moves outside the circle. Suppose that you have a segment $\overline{PB}$ intersecting a circle at A and B and a second segment $\overline{PD}$ intersecting the circle at C and D. Then is it still true that

$$PA \cdot PB = PC \cdot PD?$$

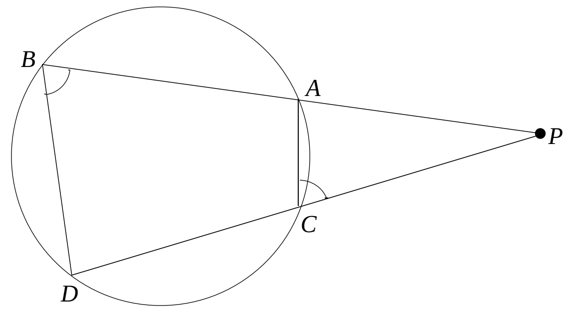
Add segments $\overline{AC}$ and $\overline{BD}$ to your drawing. If you can show that $\triangle PAC$ and $\triangle PDB$ are similar, you'll be done, since you can then conclude that

$$\frac{PA}{PD} = \frac{PC}{PB},$$

implying that

$$PA \cdot PB = PC \cdot PD.$$

USING SIMILARITY (continued)



First, notice that the triangles share $\angle BPD$. You can show that

$$m\angle ABD + m\angle ACD = 180^\circ,$$

since the angles form arcs that completely cover the circle (we'll show below why this is true). Assuming this fact, it follows that

$$\angle ABD \cong \angle ACP,$$

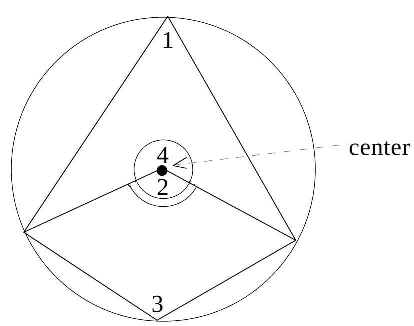
since $m\angle ACP + m\angle ACD$ also equals 180°. Therefore, by the AA triangle similarity test, you know that

$$\triangle PAC \sim \triangle PDB,$$

as desired.

Now, why is it true that the sum of the measures of angles $\angle ABD$ and $\angle ACD$ is 180°? In other words, if two angles span two arcs that together cover a circle, why is the sum of their angle measures 180°? In the picture below, we want to show

$$m\angle 1 + m\angle 3 = 180^\circ.$$



We will use the following facts about measures of angles in a circle:

- The measure of a central angle is equal to the measure of its intercepted arc.
- The measure of an inscribed angle is one half the measure of its intercepted arc.

Then,

$$m\angle 1 = \frac{1}{2} m\angle 2$$

and

$$m\angle 3 = \frac{1}{2} m\angle 4.$$

These equations together imply that

$$\begin{aligned} m\angle 1 + m\angle 3 &= \frac{1}{2}(m\angle 2 + m\angle 4) \\ &= \frac{1}{2}(360^\circ) \\ &= 180^\circ. \end{aligned}$$

Problem 44 *(Student page 129)* You can construct a constant-perimeter rectangle in much the same way as you constructed a constant-area rectangle. Draw a segment $\overline{AB}$, along with a point C, which is allowed to move back and forth along $\overline{AB}$. Notice that as C moves, the sum $AC + BC$ remains constant.

Now, using the appropriate geometry software commands, construct two concentric circles, one with radius AC, and one with radius BC. Draw radii of these circles that are at right angles, and use them to construct a rectangle, as you did in Problem 37. Now move point C along $\overline{AB}$. The dimensions of the rectangle will adjust accordingly, but its perimeter, which is equal to $2(AC + BC)$, will be invariant.

The Geometric Mean

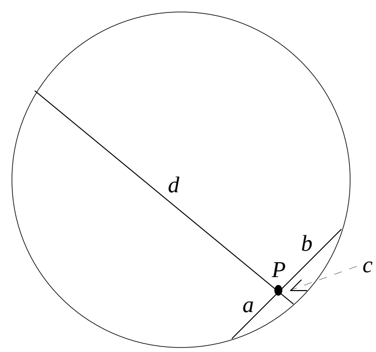
Problem 45 *(Student page 130)* Follow the instructions in the Student Module to obtain missing length d. Why does this work? Because the power of a point is constant, you know that

$$a \cdot b = c \cdot d,$$

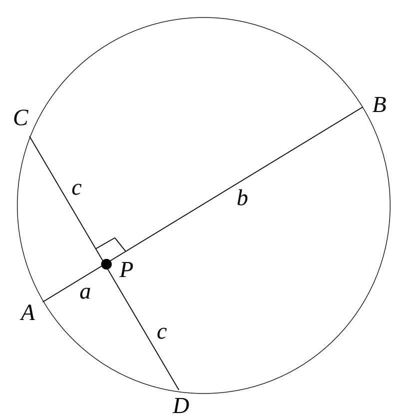
which means that if you construct a rectangle with length c and height d, it will have the same area as the original $a \times b$ rectangle.

How do you construct the unique circle passing through three given points?

Problem 46 *(Student page 131)* Once again, draw segments of lengths a and b, joining them at P to form a segment of length $a + b$. Now draw another segment with its endpoint at P and having length c. You now have three free endpoints, and there is a unique circle passing through them. Construct this circle, and extend the segment of length c so that it intersects the opposite side of the circle, forming a segment of the desired length, d.



Problems 47–49 *(Student pages 132–133)* Once again, draw segments a and b so that they meet at a point P to form a longer segment, $\overline{AB}$. Now draw a circle with $\overline{AB}$ as a diameter, and draw a chord $\overline{CD}$ through P perpendicular to the diameter:



USING SIMILARITY (continued)

Since $\overline{AB}$ is a diameter of the circle,

$$PC = PD = c.$$

Applying the Power-of-a-Point Theorem gives

$$AP \cdot PB = PC \cdot PD$$

or equivalently,

$$ab = c^2.$$

Thus, c is the length of a square with area ab.

Problem 50 *(Student page 133)* The geometric means are 4, 6, $2\sqrt{6}$, and 5, respectively.

Problem 51 *(Student page 135)* Any triangle inscribed in a semicircle is a right triangle.

Problem 52 *(Student page 135)* In both figures, BD is the geometric mean of AD and DC, so you know that

$$(BD)^2 = AD \cdot DC.$$

In the first figure,

$$(BD)^2 = 2 \cdot 3,$$

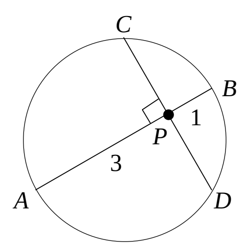
so $BD = \sqrt{6}$. Applying the Pythagorean Theorem to $\triangle ADB$ and $\triangle CDB$ gives $AB = \sqrt{10}$ and $BC = \sqrt{15}$.

For the second figure, use the Pythagorean Theorem to calculate that $BD = \sqrt{3}$. Then,

$$(\sqrt{3})^2 = AD \cdot 1,$$

so $AD = 3$. You can then calculate $AB = \sqrt{12}$.

Problem 53 *(Student page 135)* Construct a circle with a diameter $\overline{AB}$ of length $3 + 1 = 4$ inches. Then draw a chord perpendicular to $\overline{AB}$ that splits it into segments of lengths 1 and 3 inches. For reasons explained in the solution for Problem 47, $\overline{PC}$ is the geometric mean of 1 and 3.



Problem 54 *(Student page 136)* Because

$$m\angle BAD + m\angle ABD = 90^\circ$$

and

$$m\angle CBD + m\angle ABD = 90^\circ,$$

we know that

$$\angle BAD \cong \angle CBD.$$

And since $\triangle ADB$ and $\triangle BDC$ are both right triangles, by the AA similarity test,

$$\triangle ADB \sim \triangle BDC.$$

Thus

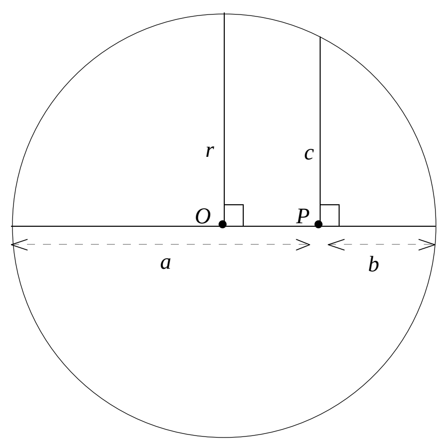
$$\frac{AD}{DB} = \frac{BD}{DC}$$

and

$$(BD)^2 = AD \cdot DC.$$

Problem 55 *(Student page 136)*

a. The given segment of length $a + b$ is a diameter of the circle. Construct c, the geometric mean of a and b as usual: draw a chord through P that intersects the diameter perpendicularly:



Notice that any radius of the circle will have length $r = \frac{a+b}{2}$, and is thus the arithmetic mean of a and b.

In particular, draw the radius through the center, O, parallel to segment of length c. As you can see in the picture above, $r > c$, so the arithmetic mean of a and b is larger than their geometric mean. Only when $a = b$ (making P the center of the circle) does the arithmetic mean equal the geometric mean.

b. The perimeter of the square is

$$4\left(\frac{a+b}{2}\right) = 2(a+b),$$

which is also the perimeter of the rectangle. The square has area $\left(\frac{a+b}{2}\right)^2$, which is simply the square of the arithmetic mean of a and b. The area of the rectangle is ab, which is the square of the geometric mean of a and b.

Since we previously showed that the arithmetic mean is greater than the geometric mean when a does not equal b, it follows that the square of the arithmetic mean is greater than the square of the geometric mean. Thus, the area of the square is larger than the area of the rectangle.

This shows that, of all rectangles with a fixed perimeter, the square is the one that encloses the most area.

Problem 56 *(Student page 137)* Each right triangle is drawn between two parallel lines, and the distance between these lines is the same in each frame. Thus, the height BD never changes. In each frame, this height is the geometric mean of AD and DC. Since BD is invariant, the quantity $(BD)^2$ is certainly also invariant, implying that the product $AD \cdot DC$ remains unchanged.

Problem 57 *(Student page 137)* Start with the same setup as in the previous problem, and construct two concentric circles, one with a radius of $\overline{AD}$ and one with a radius of $\overline{DC}$. Draw the radii perpendicular to each other, creating a rectangle that has sides congruent to these two radii. As you drag or animate point D along its line, the product $AD \cdot DC$ doesn't vary, and this product is the area of the rectangle you constructed. Thus, by moving D, you obtain a sequence of constant-area rectangles.

Problem 58 *(Student page 138)* In the power-of-a-point construction, the range of rectangles you can obtain is limited by your choice of the circle and of the point P inside the circle. In the geometric mean construction, you can obtain a much larger range of rectangles, since the length of $\overline{AD}$ becomes infinitely small as point D approaches point A.

Problem 59 *(Student page 138)*

a. First, look at $\triangle PDE$, noticing that CD is the geometric mean of PC and CE. Apply the Pythagorean Theorem to $\triangle PCD$ to find CD:

$$1^2 + (CD)^2 = x^2,$$

so $(CD)^2 = x^2 - 1$. Now, using the geometric mean formula,

$$(CD)^2 = PC \cdot CE,$$

implying that

$$x^2 - 1 = 1 \cdot CE,$$

so

$$CE = x^2 - 1.$$

Thus,

$$PE = PC + CE = 1 + (x^2 - 1) = x^2.$$

b. Now look at $\triangle PEF$. You'll see that DE is the geometric mean of PD and DF. Apply the Pythagorean Theorem to $\triangle PED$:

$$x^2 + (DE)^2 = (x^2)^2,$$

so $(DE)^2 = x^4 - x^2$. Then, the geometric mean formula says that

$$(DE)^2 = PD \cdot DF,$$

so

$$x^4 - x^2 = x \cdot DF.$$

Factor on the left to obtain

$$x(x^3 - x) = x \cdot DF.$$

Because x represents a length, we know that x is a positive number. Since we are sure that $x \neq 0$, we may divide both sides by x to get

$$DF = x^3 - x.$$

Therefore,

$$PF = PD + DF = x + (x^3 - x) = x^3.$$

c.–d. Continue in this manner, each time using a right triangle to make a geometric mean statement. The remaining answers are:

- $PG = x^4$
- $PH = x^5$
- $PI = x^6$
- $PJ = x^7$
- $PK = x^8$.

These lengths form a *geometric sequence* — each length is obtained from the previous length by multiplying by x.

No Measuring, Please!

Problem 60 *(Student page 139)* To form a segment of length $a + b$, simply line up the segment of length a end-to-end with the segment of length b.

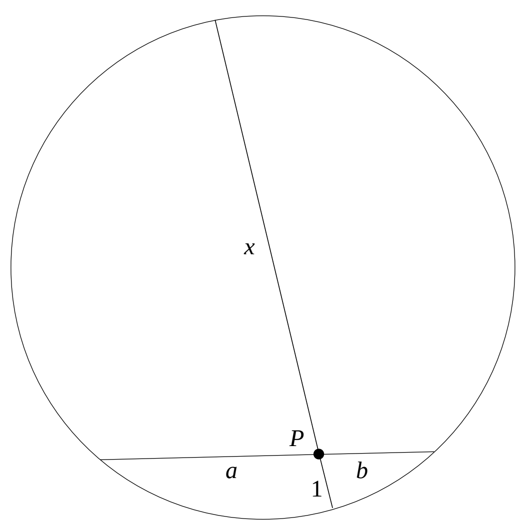
Problem 61 *(Student page 139)* Draw a segment of length a, and use a compass to mark off a piece of this segment of length b. The remainder will have length $a - b$.

Problem 62 *(Student page 139)* Lay three copies of the length a segment end-to-end to produce a segment of length $3a$.

Problem 63 *(Student page 139)* Lay two copies of the length a segment next to each other to make a segment of length $2a$. Now proceed as in Problem 61 to produce a segment of length $2a - b$.

Problem 64 *(Student page 139)* You can use a technique from the section "The Geometric Mean" to solve this problem. Lay the length a and the length b segments next to each other so that they meet at a point P, forming a segment of length $a + b$. Then draw a unit segment that has P as one of its endpoints. Draw a circle that passes through each of the three free endpoints, as shown below, and extend the unit segment so that it intersects the circle at a second point, forming a segment of length x:

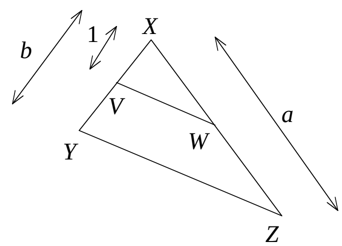
Note: In all the pictures at the right and on the following pages, we shortened the lengths of *a*, *b*, and 1 so that the pictures would not become too large.



Then, using what you know about the power of point P, you see that $a \cdot b = x \cdot 1$, so that $x = ab$.

A question: Can you solve this problem by constructing a pair of similar triangles instead?

Problem 65 *(Student page 139)* Draw a triangle XYZ with side lengths a and b (the other side length does not matter). On the side of length b, mark off the unit length. Then draw a segment $\overline{VW}$ through point V parallel to side $\overline{YZ}$:



Using the Parallel Theorem,

$$\frac{1}{b} = \frac{XW}{a}$$

so

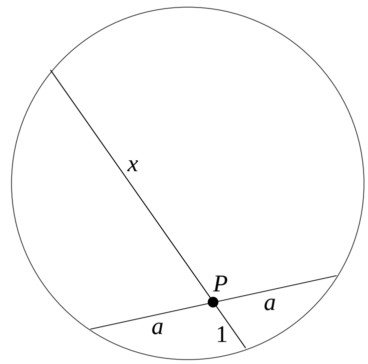
$$XW = \frac{a}{b}.$$

Problem 66 *(Student page 139)* Use notebook paper and the parallels method from the section "Segment Splitters" to divide the segment of length b into three congruent pieces, thereby creating a segment of length $\frac{b}{3}$.

Can you also figure out a way to solve this problem by using the technique from Problem 65?

Problem 67 *(Student page 139)* This is similar to Problem 64. Lay two copies of the segment of length a end-to-end to form a segment of length $2a$, and let P be the midpoint of this segment. Draw a unit segment that has P as one of its endpoints.

Then construct a circle passing through the three free endpoints, and extend the unit segment so that it intersects this circle at a second point, forming a segment of length x:



Now use what you know about the power of point P:

$$x \cdot 1 = a \cdot a,$$

so

$$x = a^2,$$

and you have constructed a segment of length a^2.

Problem 68 *(Student page 139)* Notice that

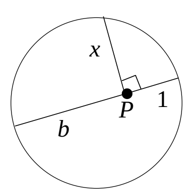
$$\sqrt{b} = \sqrt{b \cdot 1}.$$

This means that $\sqrt{b}$ can be thought of as the geometric mean of b and 1. So construct a segment of length $b + 1$ by laying two segments of length b and 1 next to each other, letting P be the point where they meet. Construct a circle having this segment as a diameter, and then draw a segment perpendicular to the first that passes through P. This segment of length x satisfies

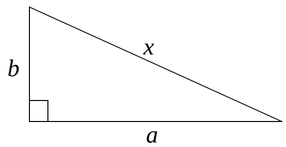
$$x^2 = b \cdot 1,$$

so

$$x = \sqrt{b}.$$



Problem 69 *(Student page 140)* The quantity $\sqrt{ab}$ is the geometric mean of a and b. Use the method from the previous problem to construct their geometric mean.



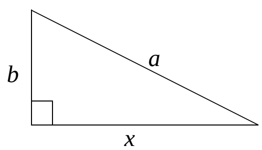
Problem 70 *(Student page 140)* Construct a right triangle with legs of length a and b. By the Pythagorean Theorem, if x is the length of the hypotenuse, then

$$x^2 = a^2 + b^2,$$

so

$$x = \sqrt{a^2 + b^2},$$

which is the desired segment length.



Problem 71 *(Student page 140)* This time, construct a right triangle with one leg of length b, and a hypotenuse of length a. Call the length of the second leg x. Then you know that

$$x^2 + b^2 = a^2,$$

so it follows that

$$x^2 = a^2 - b^2,$$

implying

$$x = \sqrt{a^2 - b^2},$$

which is the length you want to construct.

Problem 72 *(Student page 140)* Sorry, the authors couldn't resist . . . this one is sneaky! Notice that

$$\frac{a+\sqrt{b}}{2} + \frac{a-\sqrt{b}}{2} = \frac{(a+\sqrt{b})+(a-\sqrt{b})}{2} = \frac{2a}{2} = a.$$

This question is really just asking you to draw a segment of length a!

Areas of Similar Polygons

Problem 1 *(Student page 141)* Remember that to scale a rectangle by a factor of 2 means to draw a new rectangle with sides twice as long as the corresponding sides of the original. So if you scale a rectangle with sidelengths b and h by 2, you will get a rectangle with sidelengths $2b$ and $2h$. Four copies of the original rectangle fit inside this scaled copy; this shows that the new rectangle has four times the area of the original. You can also verify this algebraically, as the area of the scaled copy is equal to $2b \cdot 2h$, which is equal to $4 \cdot bh$, where bh is the area of the original rectangle.

Problem 2 *(Student page 141)* If you take a rectangle with dimensions b by h and scale it by a factor of $\frac{1}{3}$, you will obtain a new rectangle with dimensions $\frac{b}{3}$ and $\frac{h}{3}$. Nine copies of this scaled figure fit inside the original, so the area of the scaled rectangle is one ninth the area of the original one.

Problem 3 *(Student page 141)* Here is one way to state the theorem:

> If rectangle $ABCD$ is scaled by a factor of r to get a rectangle $A'B'C'D'$, then the area of $A'B'C'D'$ is r^2 times the area of $ABCD$.

Notice that if the scaling factor, r, is less than 1, the scaled rectangle will have a smaller area than the original.

To see this, let b and h be the sidelengths of rectangle $ABCD$. Then the scaled copy $A'B'C'D'$ will have sidelengths rb and rh. Thus, the area of $A'B'C'D'$ is

$$rb \cdot rh = r^2 \cdot bh.$$

Since the area of $ABCD$ is bh, it follows that the area of the scaled copy is r^2 times the area of the original.

Problem 4 *(Student page 141)* Suppose that you start with a triangle that has sidelengths a, b, and c, and scale it by a factor of r. The new triangle will have sidelengths ra, rb, and rc.

a. The ratio of the perimeters will be r. The perimeter of the original triangle is

$$a + b + c,$$

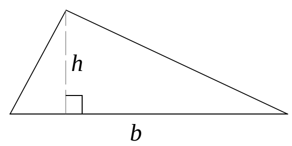
and perimeter of the scaled copy is

$$ra + rb + rc = r(a + b + c),$$

so the ratio of the new perimeter to the old one is

$$\frac{r(a+b+c)}{a+b+c} = r.$$

b. See the notes for Problem 20 of Investigation 4.15 in this Solution Resource.



c. Let h be the length of the altitude drawn from side b of the original triangle. Then the corresponding lengths in the scaled triangle are rh and rb. Thus, the ratio of the area of the scaled triangle to the area of the original triangle is

$$\frac{\frac{1}{2}rb \cdot rh}{\frac{1}{2}bh} = r^2.$$

Problem 5 *(Student page 141)* If the sides of a triangle are tripled, then the area will become 9 times as large. The original triangle has an area of

$$\frac{1}{2} \cdot 10 \cdot 12 = 60,$$

so the new triangle has an area of 540.

Problem 8 *(Student page 142)*

a. Since Polygon 1 was scaled by a factor of r, each of the four triangles was also scaled by a factor of r. This means that their areas change by a factor of r^2, so the corresponding triangles in Polygon 2 have areas of r^2a, r^2b, r^2c, and r^2d.

b. The total area of Polygon 2 is the sum of these areas:

$$r^2a + r^2b + r^2c + r^2d = r^2(a + b + c + d).$$

c. Since the area of Polygon 1 is

$$a + b + c + d,$$

this shows that when Polygon 1 is scaled by a factor of r, its area changes by a factor of r^2.

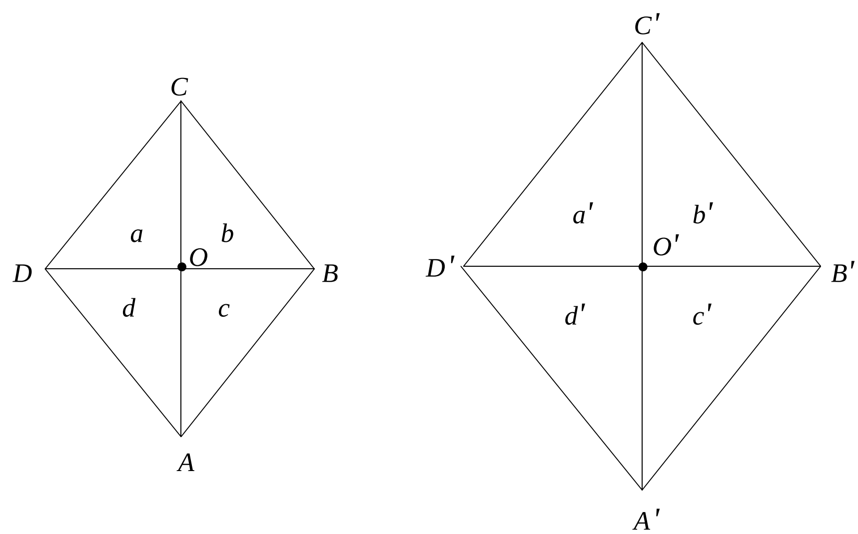
Problem 9 *(Student page 143)* Suppose $ABCD$ is scaled by r to get $A'B'C'D'$. Pick a point O inside $ABCD$ and connect it to the vertices, dividing $ABCD$ into triangles (four in the picture on the next page but as many as there are sides in general). Let the areas of these triangles be a, b, c, and d. Then

$$\text{Area}(ABCD) = a + b + c + d.$$

If the image of O is O', then triangle DOC gets scaled by a factor of r to triangle $D'O'C'$, and so on. So, by Problem 4 on page 141, the area of the triangles in $A'B'C'D'$

are r^2a, r^2b, r^2c, and r^2d. Then

$$\begin{aligned}\text{Area}(A'B'C'D') &= r^2a + r^2b + r^2c + r^2d \\ &= r^2(a+b+c+d) \\ &= r^2 \cdot \text{Area}(ABCD).\end{aligned}$$



Problem 10 *(Student page 143)* Each dimension of the $200' \times 300'$ cornfield is one half the corresponding dimension of the $400' \times 600'$ cornfield, implying that, if you scale the larger field by a factor of $\frac{1}{2}$, you will obtain the smaller field. From what you know about area, this means that the area of the smaller field is one fourth the area of the larger field. Since the big field takes eight bags of seed, the small field should only take two bags of seed, since $\frac{1}{4} \cdot 8 = 2$. Bessie sold Hans too many bags!

Problem 11 *(Student page 143)* If you scale the polygon by a factor of 1.5, the new area will be $(1.5)^2 = 2.25$ times the area of the original polygon.

Problem 12 *(Student page 143)*

a. If one square has an area that is 12 times the area of another square, then the ratio of their sides is $\sqrt{12}$. Since all squares are scaled copies of each other, the ratio of their areas is equal to the square of the ratio of their sides.

To see this algebraically, let r and s be the sidelengths of the two squares.

You are given that

$$s^2 = 12r^2,$$

and want to solve for the value of $\frac{s}{r}$.

$$\frac{s^2}{r^2} = 12,$$

so taking the positive square root of both sides gives

$$\frac{s}{r} = \sqrt{12}.$$

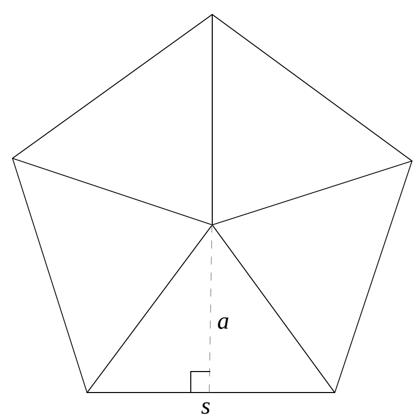
b. Recall that if a square has sides of length s, its diagonals have length $s\sqrt{2}$. So, in this case, the ratio of the diagonals is

$$\frac{s\sqrt{2}}{r\sqrt{2}} = \frac{s}{r} = \sqrt{12}.$$

Therefore, the ratio of the diagonals is the same as the ratio of the sides.

Problem 13 *(Student page 144)*

a. Take any regular polygon and draw segments from the polygon's center to each vertex; this will divide the polygon into n congruent triangles, where n is the number of polygon sides. Any apothem you draw will be an altitude of one of the triangles. Since the triangles are congruent, the altitudes (apothems) will all be congruent.



b. Let s be the length of each side of the polygon. Thus, each triangle has area

$$\frac{1}{2}sa,$$

where a is the length of an apothem of the polygon.

The total area of the polygon is the sum of the areas of these triangles. Since there are n congruent triangles, this total area is given by

$$A = n \cdot \frac{1}{2}sa.$$

But the perimeter, P, of the polygon can be expressed as

$$P = ns,$$

so you can rewrite the area, A, of the polygon as

$$A = \frac{1}{2}Pa.$$

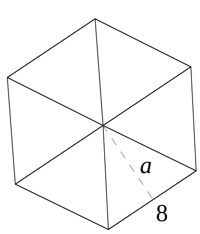
Problem 14 *(Student page 144)* The length of an apothem of a square is simply equal to one half its sidelength. So for a square whose sidelength is 12, you have $a = 6$ and $P = 48$. Therefore, the area of the square is

$$A = \frac{1}{2} \cdot 48 \cdot 6 = 144.$$

This agrees with the answer we get by calculating the area of the square in the standard way:

$$\begin{aligned} A &= s^2 \\ &= 12^2 = 144. \end{aligned}$$

Problem 15 *(Student page 144)* A regular hexagon with sidelength 8 has a perimeter $P = 6 \cdot 8 = 48$. To find its area, you need to know the apothem length.



Draw the six interior triangles formed by connecting the center of the hexagon to its vertices. These will be isosceles triangles, since all the segments from the center to the vertices have the same length. Notice also that the central angle of each triangle is equal to one sixth of a full circle, or 60°. An isosceles triangle with a 60° vertex angle must be an equilateral triangle, since the remaining two angles will also be 60°. Therefore, the apothem is equal to the altitude in an equilateral triangle with sidelength 8, so it has a length of $4\sqrt{3}$.

Applying the area formula gives

$$A = \frac{1}{2}Pa = \frac{1}{2} \cdot 48 \cdot 4\sqrt{3} = 96\sqrt{3}.$$

Problem 16 *(Student page 144)* If a rectangle is scaled by a factor of $\frac{1}{4}$, its area decreases by a factor of $\frac{1}{16}$.

Problem 17 *(Student page 144)* If a triangle is scaled by a factor of 5, its area increases by a factor of 25.

Problem 18 *(Student page 144)* If a polygon with an area of 17 square inches is scaled by a factor of 2, its new area is $17 \cdot 2^2$ square inches, or 68 square inches.

Problems 19–20 *(Student pages 145–146)* This is a hands-on activity that lets you use what you know about the relationship between scaling and area. To solve this problem, decide on a scaling factor that shrinks each piece of land to a manageable size, scale each piece of land appropriately, and compute the area of the scaled figures. Then use the fact that when you scale a polygon by an amount r, the area is scaled by a factor of r^2 to find the area of the actual pieces of land.

You might scale your figure differently than we do.

One of the larger measurements given is about 800 feet. A reasonable measurement to fit on a piece of paper or on a computer screen is 2 inches. Let's find a scale that sends 2 inches to 800 feet. We want the units to agree, so think of it as scaling 2 inches to 9600 inches (since 800 feet equals 9600 inches). The scaling factor is

$$\frac{9600 \text{ inches}}{2 \text{ inches}} = 4800.$$

This shows that 1 inch on the scaled copy will represent 4800 actual inches; put differently, 1 inch on the scaled copy represents 400 feet.

Let's scale the first parcel of land. We do this by taking each measurement individually and scaling it appropriately. Start with the first measurement of 798 feet. Since 1 inch corresponds to 400 feet, we set up the following proportion:

$$\frac{1 \text{ inch}}{400 \text{ feet}} = \frac{y \text{ inches}}{798 \text{ feet}},$$

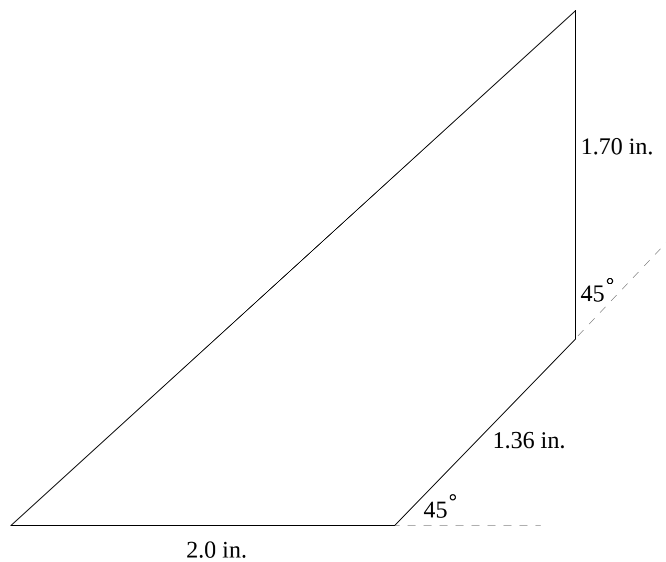
where y is the length of the segment on the scaled figure that corresponds to the actual length of 798 feet.

So

$$y = \frac{798}{400} = 1.995.$$

Let's round this number up and say that the length of 798 feet corresponds to 2.00 inches on our drawing. Doing this with the remaining measurements, we see that 543 feet corresponds to $\frac{543}{400}$ inches, and 678 feet corresponds to $\frac{678}{400}$ inches. After rounding, we obtain rounded values of 1.36 inches and 1.70 inches, respectively.

The scaled copy of Parcel 1 looks like this (remember that scaling preserves angle measurements):



Scaled Picture of Parcel 1

The area of this figure is approximately 3.5 square inches, which you can compute with geometry software. If you are working on paper, you can divide the figure into triangles and then calculate the area of each triangle.

So what is the area of the actual piece of land? Our calculations are in inches, so to get from our scaled copy back to the original, we need to scale by 4800. This has the effect of changing the area by a factor of 4800^2, which equals 23,040,000. So the total area of the land parcel is

$$3.5 \times 23{,}040{,}000 = 80{,}640{,}000 \text{ in}^2.$$

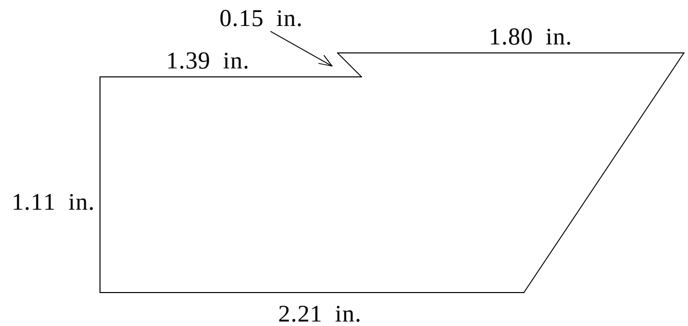
To find the area in square feet, divide by 144, getting an area of 560,000 ft^2.

Why do we divide by 144 to convert from square inches to square feet? Why not divide by 12?

AREAS OF SIMILAR POLYGONS (continued)

Now repeat this process for the second parcel of land, using the same scaling factor. Divide by 400 to see that the measurements of 884 ft, 442 ft, 554 ft, 61 ft, and 718 ft correspond to 2.21 in, 1.11 in, 1.39 in, 0.15 in, and 1.80 in, respectively.

The scaled copy of Parcel 2 looks like this:



Scaled Picture of Parcel 2

This polygon has an area of approximately 3.1 square inches. To find the actual land area, multiply by 23,040,000 to get

$$3.1 \times 23{,}040{,}000 = 71{,}424{,}000 \text{ in}^2 = 496{,}000 \text{ ft}^2.$$

We can finally conclude that, if total area is our only consideration, the recreation center should be built on the first parcel of land.

Note: If you use a different scaling factor, your answers will be different than ours — possibly *quite* different. Since we're scaling by such a large amount, small amounts of rounding cause substantial variations in the final answers. This is the disadvantage of drawing such a small-scale model.

Areas of Blobs and Circles

Problem 1 *(Student page 149)* The estimate can be improved by using a grid consisting of smaller squares and repeating the process.

Problem 2 *(Student page 149)* Each little square has an area of $\frac{1}{16}$ square inch.

Problem 3 *(Student page 149)* If A is the area of the blob, you can calculate that

$$\frac{33}{16} < A < \frac{77}{16},$$

since there are 33 squares completely contained inside the blob, and 77 squares that are either contained in or touch the blob. In other words,

$$2.0625 < A < 4.8125.$$

The first estimate was

$$1.25 < A < 6.75.$$

This new estimate is much better since

$$4.8125 - 2.0625 = 2.75,$$

while

$$6.75 - 1.25 = 5.5.$$

Thus, you've greatly narrowed down the range for the area, A. Using smaller squares gives better estimates of the blob's area.

Note: As the squares get smaller, it becomes more difficult to accurately determine which squares are inside or touch the blob. Your answers for Problems 3 and 5 may vary slightly from those shown here.

Problem 4 *(Student page 150)* Each little square now has an area of $\frac{1}{64}$ square inch.

That's a lot of squares to count!

Problem 5 *(Student page 150)* In calculating the inner sum, you find about 183 squares completely contained within the blob, giving a lower estimate of $\frac{183}{64}$ for the area. In calculating the outer sum, you'll find about 260 squares that are either contained in or touch the blob, giving an upper estimate of $\frac{260}{64}$ for the area. Therefore,

$$\frac{183}{64} < A < \frac{260}{64},$$

or

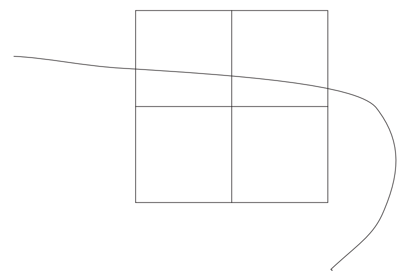
$$2.8594 < A < 4.0625.$$

Again, this is a better estimate than before, since

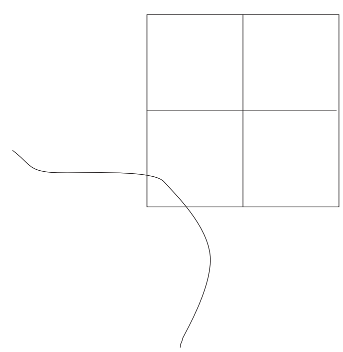
$$4.0625 - 2.8594 = 1.2031.$$

You now have A contained inside an even smaller interval.

Problem 6 *(Student page 150)* If you make the mesh finer, some parts that were not originally counted in the inner sum because they were part of a "touching" square could be completely inside the figure. For example, in the picture of one piece of the blob, none of the larger square would be counted in the inner sum using the large mesh, but two of them would be counted using the finer mesh.



If you make the mesh finer, some parts that were counted in the outer sum because they were part of a "touching" square could be completely outside the figure. For example, in the picture of one piece of the blob, the whole square would be counted using the large mesh, but only one fourth of it would be counted using the finer mesh.



So, as you make a finer mesh, the inner sum gets bigger, and the outer sum gets smaller. They never pass each other because the inner sum is always smaller than the blob and the outer sum is always larger than the blob. So they must get closer to each other (the difference between them gets smaller).

Problem 7 *(Student page 151)* Because a circle is symmetric, you can use the mesh to approximate the area of half the circle, and then multiply by 2 to estimate the total area (or use the mesh to approximate the area of one fourth the circle, and multiply by 4). As you take finer meshes, your area estimates should be approaching a value a little bit bigger than 3 square feet.

Problem 8 *(Student page 152)* Since the rubber sheet was stretched uniformly by a factor of r, the area of each square increases by a factor of r^2. The blob's area can be approximated by the sum of the areas of these squares. If the sum of the squares before they were stretched was S, the sum becomes r^2S after stretching. Thus the area of the blob has increased by a factor of approximately r^2.

In fact, if a blob is scaled by r, its area increases by a factor of *exactly* r^2 (the same result as with polygons). This can be shown by formalizing the argument above.

Problem 9 *(Student page 153)* From the previous problem, it seems that if any blob is scaled by r, its area changes by r^2. Thus, since the figure on the right was obtained by scaling the figure on the left by 2, its area should be 4 times as much. In other words, its area is 16 square inches.

Problem 10 *(Student page 153)* The diameter of the first moon is 4.5 cm, and the width of the second moon is 3.5 cm. The scaling factor (from the first to the second) is $\frac{3.5}{4.5} \approx 0.78$, so the ratio of their areas is about $(0.78)^2 = 0.6084$.

Problem 12 *(Student page 154)* The exact area of a 3–4–5 right triangle is 6.

Perimeters of Blobs and Circles

Problems 1–4 *(Student page 156)* You can estimate the perimeter of a blob by approximating its boundary with line segments and adding their lengths. To get a better approximation, use shorter segments. By using shorter segments, you better approximate the curve, giving a better estimate of its perimeter. Moreover, you can improve the estimate as much as you want by taking a large enough number of very small segments.

Problems 5–7 *(Student pages 157–159)* As the number of polygon sides increases, the difference between the outer and inner perimeters gets smaller because the perimeters of the inscribed and circumscribed polygons get closer and closer to the circle's circumference.

Problem 8 *(Student page 159)*

Number of Sides	Outer Perimeter	Inner Perimeter	Difference
4	21.2 cm	13.2 cm	8.0 cm
8	17.6 cm	16.0 cm	1.6 cm
16	16.8 cm	16.0 cm	0.8 cm

Problem 9 *(Student page 159)* The circle's perimeter is clearly between 16 cm and 16.8 cm. The average of the upper and lower bounds is not the true perimeter, but it gives a reasonable estimate of 16.4 cm.

Problem 10 *(Student page 162)* Imagine that each of the polygons is inscribed in a circle. As the number of polygon sides increases, the length of the apothem approaches the radius, r, of the circumscribed circle and the polygon's perimeter approaches the circumference of this circle ($2\pi r$). Thus, the value of the perimeter divided by the length of the apothem approaches $\frac{2\pi r}{r} = 2\pi$, which is approximately 6.28.

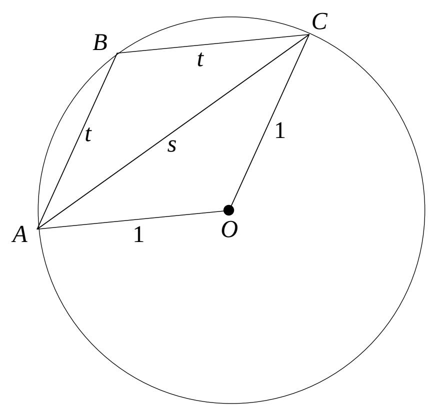
Problem 11 *(Student page 162)* As the logs make one revolution, the horizontal distance that they travel is the same as their circumference, 6 feet. It's tempting, then, to say that the house moves 6 feet. But in fact, the house moves more than that. As the house is pushed, it slides along the logs. It slides 6 feet, since this is how far the logs move. Thus, the total distance the house moves is the sum of the distances traveled by the logs and by the house sliding along the logs. The total distance thus is 12 feet.

Problem 12 *(Student page 162)* Let s be the sidelength of the regular polygon with n sides inscribed in a circle of radius 1. Let t be the sidelength of the regular

polygon with $2n$ sides inscribed in the same circle. You want to show that

$$t = \sqrt{2 - \sqrt{4 - s^2}}.$$

Notice that in each polygon, the vertices are evenly spaced along the circle. Suppose you have already drawn the polygon with n sides. Take two consecutive vertices, A and C, and look at the arc of the circle they span. Find the midpoint of this arc, and call it B. Then A, B, and C will be consecutive vertices of the polygon with $2n$ sides. By adding new vertices in between the "old" ones in this way, you create the $2n$ vertices you need for the new polygon:



Now draw segment $\overline{OB}$. Because we are working with evenly-spaced points on the circle, this segment bisects $\angle AOC$, so

$$m\angle AOB = m\angle COB.$$

Let D be the point of intersection of $\overline{OB}$ with the side $\overline{AC}$. By the SAS congruence test,

$$\triangle AOD \cong \triangle COD.$$

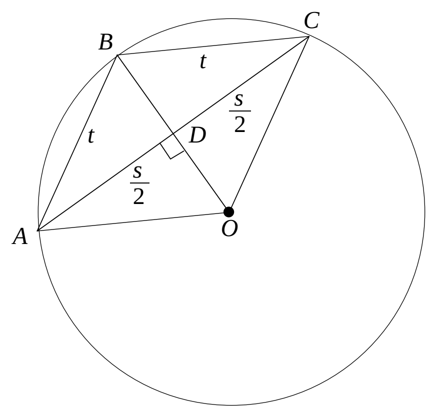
This means that

$$AD = CD = \frac{s}{2}$$

and

$$m\angle ADO = m\angle CDO = 90^\circ.$$

PERIMETERS OF BLOBS AND CIRCLES (continued)



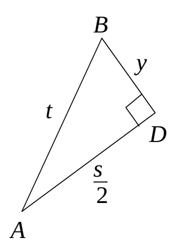
Now look at $\triangle OAD$, and let $x = OD$. By the Pythagorean Theorem,

$$x^2 + \left(\frac{s}{2}\right)^2 = 1,$$

so

$$x = \sqrt{1 - \frac{s^2}{4}}.$$

Now look at $\triangle ADB$, and let $y = DB$. Again, apply the Pythagorean Theorem:



$$y^2 + \left(\frac{s}{2}\right)^2 = t^2$$

so

$$t = \sqrt{y^2 + \frac{s^2}{4}}.$$

Because $\overline{OB}$ is a radius, you know that

$$x + y = 1$$

or

$$y = 1 - x,$$

so

$$y = 1 - \sqrt{1 - \frac{s^2}{4}}.$$

Then,

$$\begin{aligned} y^2 &= \left(1 - \sqrt{1 - \frac{s^2}{4}}\right)^2 \\ &= 1 + \left(1 - \frac{s^2}{4}\right) - 2\sqrt{1 - \frac{s^2}{4}} \\ &= 2 - \frac{s^2}{4} - 2\sqrt{1 - \frac{s^2}{4}}. \end{aligned}$$

Substituting this expression for y^2 in the equation for t from the previous page gives

$$\begin{aligned} t &= \sqrt{\left(2 - \frac{s^2}{4} - 2\sqrt{1 - \frac{s^2}{4}}\right) + \left(\frac{s^2}{4}\right)} \\ &= \sqrt{2 - 2\sqrt{1 - \frac{s^2}{4}}} \\ &= \sqrt{2 - 2\sqrt{\frac{4 - s^2}{4}}} \\ &= \sqrt{2 - 2\left(\frac{\sqrt{4 - s^2}}{2}\right)} \\ &= \sqrt{2 - \sqrt{4 - s^2}}. \end{aligned}$$

This is exactly what you wanted to show.

CIRCLES AND 3.14159265358979323...

Problem 1 *(Student page 163)* Take the circle with radius 12 and scale it by $\frac{5}{2}$. One way to do this it to perform a dilation. Scaling the circle by $\frac{5}{2}$ will give another circle with radius 30, since all circles are similar.

Problem 2 *(Student page 163)* In general, if a circle of radius r is scaled by a factor of $\frac{R}{r}$, it will become a circle with radius R.

Problems 3–4 *(Student pages 163–164)* In Problem 7 in Investigation 4.18, you approximated the area of a circle with radius one foot to be a little bit more than 3 square feet. If you scale this circle by a factor of 2, you obtain a circle with radius 2 feet. And, since scaling a circle by r multiplies its area by r^2, it follows that a circle with radius 2 feet has an area of a bit more than $3 \cdot 2^2 = 12$ square feet. It follows that circles with radii of 5, 6, $\sqrt{3}$, or $7\frac{1}{2}$ feet will have areas that are approximately 75, 108, 9, and 169 square feet, respectively.

In general, any circle with radius r can be obtained by scaling the 1-foot radius circle by r. This multiplies the area by a factor of r^2, so you would estimate that the circle with radius r has an area of approximately $3r^2$.

Problem 5 *(Student page 165)* Henri is certainly correct in that a circle of radius 1 foot doesn't have the same area as a circle of radius 1 inch. What is true, however, is that the circle of radius 1 foot will have an area of π square *feet*, and the circle of radius 1 inch will have an area of π square *inches*.

Problem 6 *(Student page 166)* The area, A, of a circle is given by

$$A = \pi r^2,$$

where r is its radius.

a. If a circle has a radius of 10 inches, its area will be 100π square inches.

b. If a circle has a radius of 5 centimeters, its area will be 25π square centimeters.

c. A circle with a diameter of 3 feet has a radius of $\frac{3}{2}$ feet, so its area is $\frac{9}{4}\pi$ square feet.

d. The scaled circle has a radius of 10 inches, so its area is 100π square inches.

Problem 7 *(Student page 166)*

a. The area of a wedge formed by a 60° central angle is equal to one sixth the

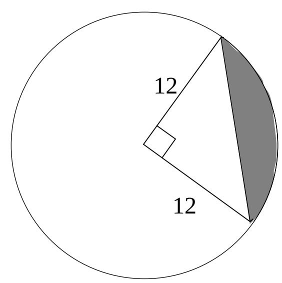
area of the circle, since 60° equals one sixth of 360°. A circle of radius 5 has an area of 25π, so the area of the wedge is

$$\frac{1}{6} \cdot 25\pi = \frac{25\pi}{6}.$$

b. This wedge is formed by a 90° central angle, so its area is one fourth the area of the circle. The circle has a radius of 10, hence an area of 100π. It follows that the area of the wedge is

$$\frac{1}{4} \cdot 100\pi = 25\pi.$$

c. The wedge of the circle is formed by a central angle of 90°. The wedge is divided into two pieces: the shaded area, and an isosceles right triangle (both legs of the triangle are radii of the circle, so they are congruent).



Since the wedge is one fourth of the circle and the radius of the circle is 12, the area of the wedge is

$$\frac{144\pi}{4} = 36\pi.$$

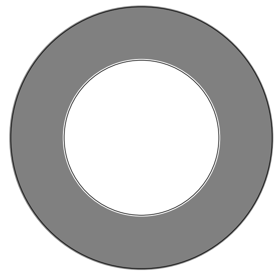
The right triangle has an area

$$\frac{1}{2} \cdot 12 \cdot 12 = 72.$$

Thus, you can subtract to see that the area of the shaded region is

$$36\pi - 72.$$

d. The area of the shaded region is equal to the area of the large circle minus the area of the smaller interior circle.



Inner radius is 4;
outer radius is 7.

The area of the large circle is 49π and the area of the small circle is 16π. It follows that the area of the shaded region is

$$49\pi - 16\pi = 33\pi.$$

Problem 8 *(Student page 168)*

a. Since

$$C = 2\pi r,$$

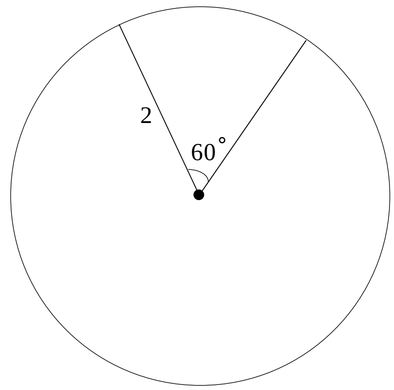
and π is approximately 3, it follows that the circumference is approximately six times the radius.

b. Since

$$C = \pi d,$$

it follows that the circumference is approximately three times the diameter.

Problem 9 *(Student page 168)*



Since 60° is one sixth of the circle, it follows that the length of the smaller arc is one sixth the circumference of the circle. The radius of the circle is 2, so

$$C = 2 \cdot \pi \cdot 2 = 4\pi,$$

implying that the small arc has length

$$\frac{4\pi}{6} = \frac{2\pi}{3}.$$

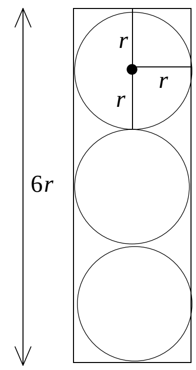
Since the two arcs together form the entire circle, you know that their lengths must add up to the circle's circumference. Therefore, the length of the longer arc is

$$4\pi - \frac{2\pi}{3} = \frac{10\pi}{3}.$$

Problem 10 *(Student page 168)* For any circle, the ratio of its circumference to its diameter is exactly π since

$$\frac{C}{d} = \frac{\pi d}{d} = \pi.$$

Problem 11 *(Student page 168)* The canister is a cylinder. Imagine a cross section of a canister containing three tennis balls:



If you call the radius of each ball r, then the height of the canister is $6r$. The circumference of the container should be equal to the circumference of any one of the balls. This circumference is equal to $2\pi r$, which is just a bit bigger than $6r$, since π is just a bit bigger than 3. Thus the circumference is larger than the height — a somewhat surprising result!

Problem 12 *(Student page 168)* This problem also involves a cylinder. The paper for the can's label will be in the shape of a rectangle. Its height will be equal to the height of the can, which is 5 inches. Its length must be equal to the circumference of the top of the can, since the label must completely wrap around the can. Since the can has a diameter of 3 inches, its circumference is 3π. Thus, the piece of paper has dimensions $5'' \times 3\pi''$.

Problem 13 *(Student page 168)* All you have to do is calculate the circumference of the tire, which will be $2\pi r$, where r is equal to the length of a spoke of your wheel.

Problem 14 *(Student page 169)* You should get the following values:

Radius	Diameter	Area	Circumference
3	6	9π	6π
$\frac{3}{2}$	3	$\frac{9\pi}{4}$	3π
$\sqrt{\frac{3}{\pi}}$	$2\sqrt{\frac{3}{\pi}}$	3	$2\pi\sqrt{\frac{3}{\pi}}$
$\frac{3}{2\pi}$	$\frac{3}{\pi}$	$\frac{9}{4\pi}$	3

CIRCLES AND 3.14159265358979323...
(continued)

Problem 15 *(Student page 169)* Let P_n be the perimeter of the regular polygon with n sides inscribed in a circle of radius 1, and let s_n be the sidelength of this polygon. By Problem 12 of Investigation 4.19, the value for s_n is given by the equation

$$s_n = \sqrt{2 - \sqrt{4 - (s_{n/2})^2}},$$

where $s_{n/2}$ is the sidelength of the polygon with $\frac{n}{2}$ sides inscribed in the same circle. So, in this problem, the Bernoulli Sisters are calculating values of s_n for

$$n = 6, 12, 24, 48, \dots .$$

Remember that as n increases, the perimeter of the regular polygon approaches the circumference of the circle, which is 2π, since the radius is 1. In symbols,

$$P_n \to 2\pi,$$

or equivalently,

$$ns_n \to 2\pi.$$

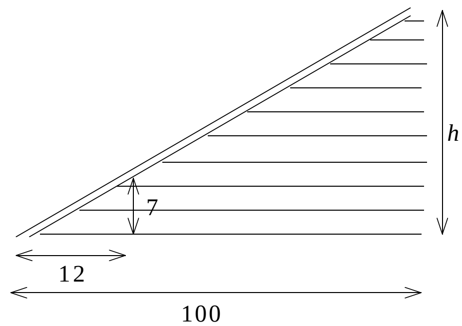
This means that

$$\frac{ns_n}{2} \to \frac{2\pi}{2} = \pi,$$

which is what you wanted to show.

So Many Triangles, So Little Time

Problem 1 *(Student page 174)* Since the roof has a 7 pitch, it rises 7 inches for every 12 inches of width. Here's a way to show this pitch on a picture of the roof for part a:



The picture in the Student Module is slightly different. Are both pictures correct? Which is more helpful?

This gives you a pair of similar triangles (why?), yielding the proportion

$$\frac{7}{12} = \frac{h}{100},$$

so

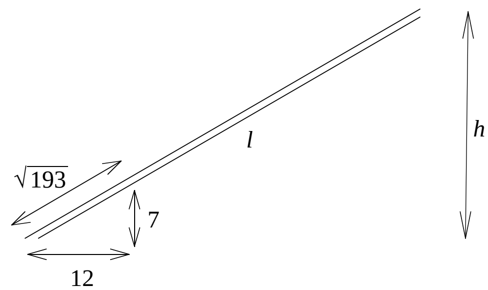
$$h = \frac{700}{12} \approx 58.3 \text{ inches.}$$

Similarly, you can calculate that widths of 150 and 210 inches correspond to heights of 87.5 and 122.5 inches, respectively.

In general, for a 7-pitch roof with a width of w and a height of h,

$$\begin{aligned} \frac{7}{12} &= \frac{h}{w} \\ 12h &= 7w \\ h &= \frac{7w}{12}. \end{aligned}$$

Problem 2 *(Student page 174)* Again you can use similar triangles. First, you need to use the Pythagorean Theorem to calculate that a right triangle with legs of lengths 7 and 12 inches has a hypotenuse of length $\sqrt{193}$:



a. When $h = 130$ inches, using similar triangles gives the proportion

$$\frac{7}{\sqrt{193}} = \frac{130}{l},$$

so

$$7l = 130\sqrt{193},$$

implying that

$$l = \frac{130\sqrt{193}}{7} \approx 258 \text{ inches.}$$

b–c. When the heights are 170 inches and 250 inches, the corresponding lengths are $\frac{170\sqrt{193}}{7} \approx 337.4$ inches and $\frac{250\sqrt{193}}{7} \approx 496.2$ inches, respectively.

d. For the general case, consider a right triangle with height h and hypotenuse l, which is similar to the 7–12–$\sqrt{193}$ right triangle. Then

$$\frac{7}{\sqrt{193}} = \frac{h}{l},$$

so

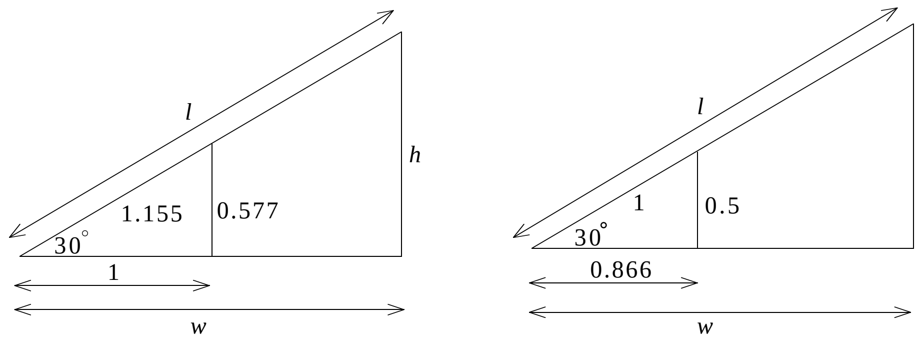
$$7l = h\sqrt{193},$$

which means that

$$l = \frac{h\sqrt{193}}{7}.$$

Any two right triangles with one other common angle are similar by the AA similarity test.

Problem 3 *(Student page 175)* Since the roof forms a right triangle with a 30° angle with the horizontal, it is similar to both of the small right triangles that are shown in the Student Module.



For each situation, you can use either triangle to solve the problem, but one of the two makes the calculations easier. Keep in mind that you want to take advantage of the unit length whenever possible.

a. If $h = 160$ inches, then use the second picture to find l:

$$\frac{160}{l} = \frac{0.5}{1},$$

so

$$l = \frac{160}{0.5} = 320 \text{ inches.}$$

b. If $w = 200$ inches, use the first picture to find h:

$$\frac{h}{200} = \frac{0.577}{1},$$

so

$$h = (200)(0.577) = 115.4 \text{ inches.}$$

c. Either triangle works easily here. If you use the first picture, then

$$\frac{l}{90} = \frac{1.155}{1},$$

so

$$l = (90)(1.155) \approx 104 \text{ inches.}$$

So Many Triangles, So Little Time
(continued)

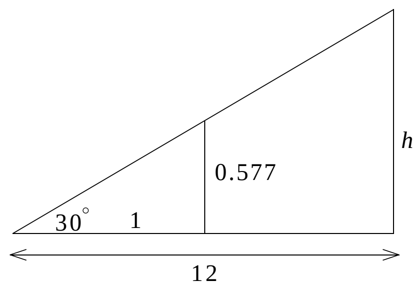
d. If $l = 250$ inches, use the second picture to find h:

$$\frac{h}{250} = \frac{0.5}{1},$$

so

$$h = (250)(0.5) = 125 \text{ inches}.$$

Problem 4 *(Student page 176)* Remember that the pitch of a roof is the height that corresponds to a width of 12 inches. For any roof with a 30° angle, you can use the following similar triangles:



Then

$$\frac{h}{12} = \frac{0.577}{1},$$

implying that

$$h = (12)(0.577) \approx 6.9.$$

Therefore, any roof with a 30° angle has a pitch of approximately 7. Notice that pitch depends on the angle of the roof and nothing else!

Problem 5 *(Student page 177)* Because the given triangles are both right triangles with the same pitch, you know they are similar.

a. Suppose $w = 210$ inches, and you want to find l:

$$\frac{w}{l} = \frac{a}{c},$$

implying that

$$\frac{210}{l} = 0.928,$$

so

$$l = \frac{210}{0.928} \approx 226.3 \text{ inches}.$$

b. Suppose $h = 300$ inches, and you want to find w:

$$\frac{h}{w} = \frac{b}{a},$$

so you know that

$$\frac{300}{w} = 0.401,$$

and

$$w = \frac{300}{0.401} \approx 748.1 \text{ inches.}$$

c. Now suppose that $l = 140$ inches, and you want to find h:

$$\frac{h}{l} = \frac{b}{c},$$

so

$$\frac{h}{140} = 0.372,$$

implying that

$$h = (0.372)(140) \approx 52.1 \text{ inches.}$$

Problem 7 *(Student page 177)* Here is part of one possible table:

Angle	$\frac{h}{w}$	$\frac{h}{l}$	$\frac{w}{l}$
25°	0.466	0.423	0.906
30°	0.577	0.500	0.866
35°	0.700	0.574	0.819
40°	0.839	0.643	0.766
45°	1.000	0.707	0.707

TRIGONOMETRY

Problem 1 *(Student page 179)* Each of the two given triangles is a right triangle with a 27° angle. Thus, the triangles are similar by AA, and the ratio $\frac{BC}{AC}$ must be the same in both.

Problem 2 *(Student page 179)* By measuring, you can calculate the value of $\frac{BC}{AC}$ to be $\frac{1}{2}$.

Problem 3 *(Student page 179)* For the first picture, you know that $AC = 40$ feet, and you want to find the height of the tree, which is BC. Use the equation

$$\frac{BC}{AC} = \frac{1}{2},$$

to see that

$$\frac{BC}{40} = \frac{1}{2},$$

so

$$BC = 20 \text{ feet.}$$

For the second picture, you are given that $BC = 300$ feet, and you want to find AC. Calculate

$$\frac{300}{AC} = \frac{1}{2},$$

so

$$AC = 600 \text{ feet.}$$

Problem 4 *(Student page 180)* Translated into the language of trigonometry, the three statements become:

a. $\sin 40^\circ = 0.64$;

b. $\cos 70^\circ = 0.34$;

c. $\tan 55^\circ = 1.43$.

Problem 5 *(Student page 180)* You should get

$$\sin A = \frac{BC}{AB} = \frac{4}{5}$$
$$\cos A = \frac{AC}{AB} = \frac{3}{5}$$
$$\tan A = \frac{BC}{AC} = \frac{4}{3},$$

and

$$\sin B = \frac{AC}{AB} = \frac{3}{5}$$
$$\cos B = \frac{BC}{AB} = \frac{4}{5}$$
$$\tan B = \frac{AC}{BC} = \frac{3}{4}.$$

Notice that

$$\sin A = \cos B$$

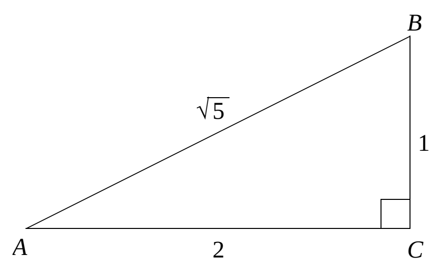
because both values equal $\frac{BC}{AB}$, and

$$\cos A = \sin B$$

because both values equal $\frac{AC}{AB}$.

Problem 6 *(Student page 181)* First, use the Pythagorean Theorem to compute the length of the hypotenuse:

$$AB = \sqrt{1^2 + 2^2} = \sqrt{5}.$$



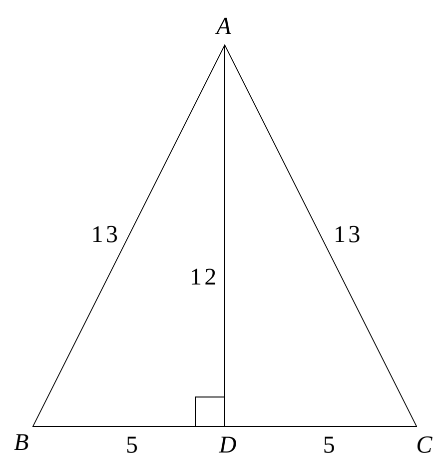
TRIGONOMETRY (continued)

Then you can see that

$$
\begin{aligned}
\sin A &= \frac{1}{\sqrt{5}}\\
\cos A &= \frac{2}{\sqrt{5}}\\
\tan A &= \frac{1}{2}\\
\sin B &= \frac{2}{\sqrt{5}}\\
\cos B &= \frac{1}{\sqrt{5}}\\
\tan B &= \frac{2}{1} = 2.
\end{aligned}
$$

Problem 7 *(Student page 181)* In order to find the trigonometric values for angle B, that angle has to be in a right triangle, which you don't have in this picture. One way to create a right triangle is to draw the altitude from vertex A. In an isosceles triangle, the altitude from the vertex angle is also a median, so $BD = CD = 5$. You can use the Pythagorean Theorem to calculate that

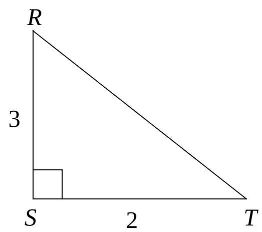
$$AD = 12.$$



Now use $\triangle DBA$ to obtain

$$\sin B = \frac{12}{13}$$
$$\cos B = \frac{5}{13}.$$

Problem 8 *(Student page 181)* Since $\tan R = \frac{2}{3}$, you can assign any values to $\overline{ST}$ and $\overline{RT}$ such that $\frac{ST}{RS} = \frac{2}{3}$. In particular, let's choose



$$ST = 2$$

and

$$RS = 3.$$

You can then calculate

$$RT = \sqrt{13}.$$

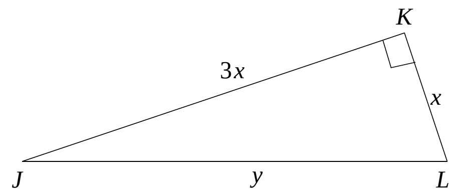
It follows that

$$\sin R = \frac{2}{\sqrt{13}}$$

and

$$\cos R = \frac{3}{\sqrt{13}}.$$

Problem 9 *(Student page 181)* If x is the length of $\overline{KL}$, then the length of $\overline{JK}$ is $3x$. Let y be the length of the hypotenuse $\overline{JL}$:



Then, by the Pythagorean Theorem,

$$\begin{aligned}(3x)^2 + x^2 &= y^2\\ 10x^2 &= y^2\\ y &= \sqrt{10}x.\end{aligned}$$

Then,

$$\sin J = \frac{x}{\sqrt{10}x} = \frac{1}{\sqrt{10}}$$
$$\cos J = \frac{3x}{\sqrt{10}x} = \frac{3}{\sqrt{10}}$$

and

$$\tan J = \frac{x}{3x} = \frac{1}{3}.$$

Also,

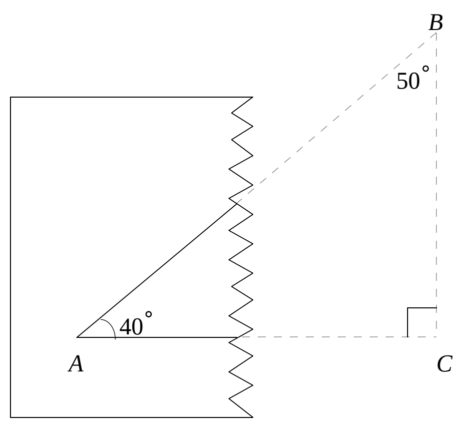
$$\sin L = \cos J = \frac{3}{\sqrt{10}}$$
$$\cos L = \sin J = \frac{1}{\sqrt{10}}$$

and

$$\tan L = \frac{3x}{x} = 3.$$

An alternate solution: You could also have solved this problem by assuming that $KL = 1$ and $JK = 3$. Why?

Problem 10 *(Student page 182)* The dashed lines below show the missing part of $\triangle ABC$:



Make sure your calculator is set in degree mode.

Using a calculator, you can compute the following values:

$$\frac{BC}{AC} = \tan 40^\circ \approx 0.84$$
$$\frac{BC}{AB} = \sin 40^\circ \approx 0.64$$
$$\frac{AC}{AB} = \cos 40^\circ \approx 0.77$$
$$\frac{AC}{BC} = \tan 50^\circ \approx 1.19.$$

The other three values listed in the problem, which are not ratios, cannot be found unless you know the side lengths of the triangle.

Problem 11 *(Student page 182)*

a. Let y be the length of the ramp. Since the ramp meets the ground 25 feet from the base of the building,

$$\cos 10^\circ = \frac{25}{y},$$

or equivalently,

$$y = \frac{25}{\cos 10^\circ} \approx 25.39 \text{ feet.}$$

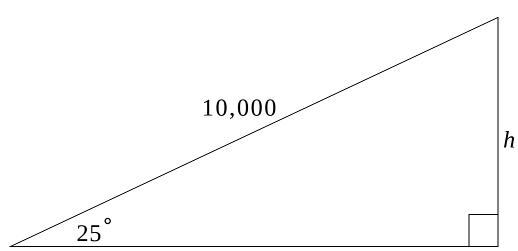
b. The side opposite the 10° angle is 2 feet long. So this time the equation you need is

$$\sin 10^\circ = \frac{2}{y}.$$

Use a calculator to find y:

$$y = \frac{2}{\sin 10^\circ} \approx 11.52 \text{ feet.}$$

Problem 12 *(Student page 183)* Let h be the airplane's height above the ground. You get a right triangle:



Then

$$\sin 25^\circ = \frac{h}{10{,}000},$$

so

$$h = 10{,}000 \sin 25^\circ \approx 4226 \text{ feet}.$$

Problem 13 *(Student page 183)* In the given picture,

$$\tan 57^\circ = \frac{BC}{1000},$$

so

$$BC = 1000 \tan 57^\circ.$$

If you estimate $\tan 57^\circ$ as 1.5, you will get

$$BC = 1500 \text{ feet},$$

and if you estimate $\tan 57^\circ$ as 1.540, you get

$$BC = 1540 \text{ feet},$$

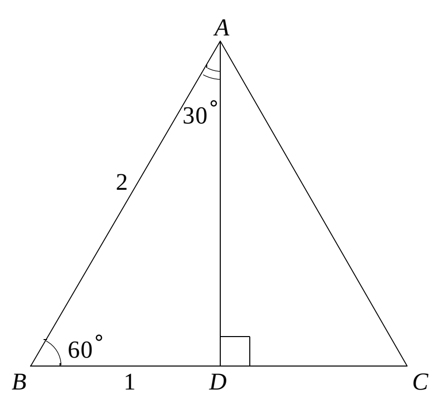
a fairly significant difference!

Problem 14 *(Student page 183)*

a. In an equilateral triangle, any altitude of the triangle is also a median, so $BD = 1$. You can now use the Pythagorean Theorem to calculate that

$$AD = \sqrt{3}.$$

.



Using $\triangle ABD$, we find that

$$\begin{aligned} \sin 60^\circ &= \frac{\sqrt{3}}{2} \\ \cos 60^\circ &= \frac{1}{2} \\ \tan 60^\circ &= \sqrt{3} \end{aligned}$$

and

$$\begin{aligned} \sin 30^\circ &= \frac{1}{2} \\ \cos 30^\circ &= \frac{\sqrt{3}}{2} \\ \tan 30^\circ &= \frac{1}{\sqrt{3}}. \end{aligned}$$

b. You can calculate that the hypotenuse of the isosceles right triangle has length $\sqrt{2}$. Thus,

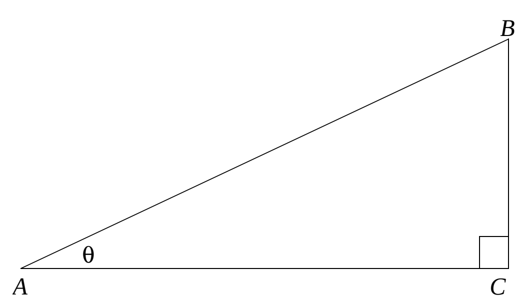
$$\begin{aligned} \sin 45^\circ &= \frac{1}{\sqrt{2}} \\ \cos 45^\circ &= \frac{1}{\sqrt{2}} \end{aligned}$$

and

$$\tan 45^\circ = 1.$$

θ (the Greek letter "theta") is a variable often used to represent angles or angle measures.

Problem 15 *(Student page 184)* The question is asking whether there is an angle θ between 0° and 90° for which $\frac{BC}{AB} = 1.5$.



Notice that $\overline{BC}$ is a leg of right triangle ABC and $\overline{AB}$ is the hypotenuse. In any right triangle, the hypotenuse is always longer than either of the legs. Thus, the denominator

of $\frac{BC}{AC}$ will always be greater than the numerator for any angle θ. So the value of $\frac{BC}{AC}$ can never be greater than one.

Problem 17 *(Student page 184)* You can use the Pythagorean Theorem to calculate that in the first picture

$$AB = \sqrt{20} = 2\sqrt{5},$$

and in the second picture

$$AC = \sqrt{7}.$$

Thus, for the first picture,

$$\sin A = \cos B = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$$
$$\cos A = \sin B = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

and

$$\tan A = 2$$
$$\tan B = \frac{1}{2}.$$

For the second picture,

$$\sin A = \cos B = \frac{3}{4}$$
$$\cos A = \sin B = \frac{\sqrt{7}}{4}$$

and

$$\tan A = \frac{3}{\sqrt{7}}$$
$$\tan B = \frac{\sqrt{7}}{3}.$$

Problem 18 *(Student page 185)* Because the sum of the angles in a triangle is 180°, you know that $m\angle BDC = 72^\circ$. Thus, $\triangle BDC$ is isosceles, with congruent sides $\overline{BD}$ and $\overline{BC}$. Therefore,

$$BC = x.$$

Since angles $\angle BAD$ and $\angle ABD$ are congruent, it follows that $\triangle ABD$ is isosceles, with congruent sides $\overline{BD}$ and $\overline{AD}$, implying

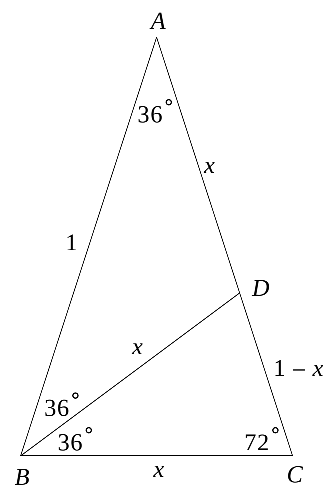
$$AD = x.$$

Notice that

$$AD + DC = AC = 1.$$

Since $AD = x$, it follows that

$$DC = 1 - x.$$



Now, by the AA similarity test, you can see that

$$\triangle ABC \sim \triangle BCD.$$

This tells us that

$$\frac{AB}{BC} = \frac{BC}{CD},$$

or

$$\frac{1}{x} = \frac{x}{1-x}.$$

Cross multiplication gives

$$x^2 = 1 - x,$$

or

$$x^2 + x - 1 = 0.$$

If you use the quadratic formula to solve this equation, you will get two values for x. However, only one is positive:

$$x = \frac{\sqrt{5} - 1}{2}.$$

You can draw the altitude of $\triangle ABC$ from A, creating a right triangle. Using this triangle, it follows that

$$\cos 72^\circ = \frac{\frac{x}{2}}{1} = \frac{x}{2} = \frac{\sqrt{5}-1}{4}.$$

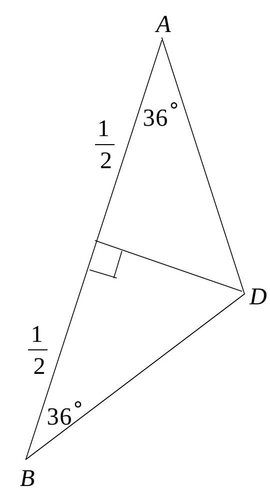
A calculator gives

$$\cos 72^\circ \approx .3090169944$$

and

$$\frac{\sqrt{5}-1}{4} \approx .3090169944.$$

Problem 19 *(Student page 185)* Draw the altitude from D in $\triangle ABD$:



It follows that

$$\cos 36^\circ = \frac{\frac{1}{2}}{x} = \frac{1}{2x} = \frac{1}{\sqrt{5}-1}.$$

A calculator gives

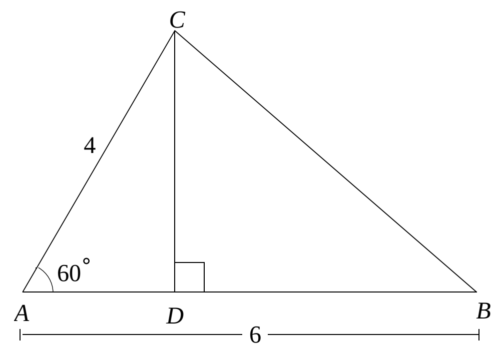
$$\cos 36^\circ \approx .8090169944$$

and

$$\frac{1}{\sqrt{5}-1} \approx .8090169944.$$

AN AREA FORMULA FOR TRIANGLES

Problem 1 *(Student page 186)* In $\triangle ABC$, draw altitude $\overline{CD}$, forming right triangle $\triangle ACD$:



The area of $\triangle ABC$ is given by the formula

$$\text{Area}\,(\triangle ABC) = \frac{1}{2}AB \cdot CD = \frac{1}{2} \cdot 6 \cdot CD = 3CD.$$

Looking at $\triangle ACD$,

$$\sin 60^\circ = \frac{CD}{4}.$$

By referring to Problem 14 of Investigation 4.22, you know that

$$\sin 60^\circ = \frac{\sqrt{3}}{2},$$

so it follows that

$$\frac{\sqrt{3}}{2} = \frac{CD}{4},$$

implying

$$CD = 2\sqrt{3}.$$

Substituting this value into the area equation above, you get

$$\text{Area}\,(\triangle ABC) = 3 \cdot 2\sqrt{3} = 6\sqrt{3}.$$

Problem 2 *(Student page 187)*

a. We'll use the method from Problem 1 to solve this one: Draw an altitude from the top vertex, and let its length be h. If A is the area of the triangle,

$$A = \frac{1}{2}bh.$$

AN AREA FORMULA FOR TRIANGLES (continued)

You want your answer to be in terms of a, b, and θ, so you need to eliminate h. The altitude forms two right triangles, and using the triangle on the left gives

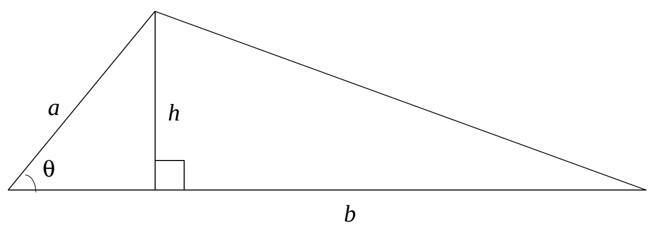
$$\sin\theta = \frac{h}{a},$$

implying that

$$h = a\sin\theta.$$

Therefore, the area of the whole triangle is given by

$$A = \frac{1}{2}ba\sin\theta.$$



b. Suppose the triangle is scaled by a factor of r. This means that every length in the triangle becomes r times the original length, so the new sides have length rb and ra. The angle θ remains the same, as scaling doesn't affect angle measurements. Using the area formula from part a, it follows that if A' is the area of the scaled triangle, then

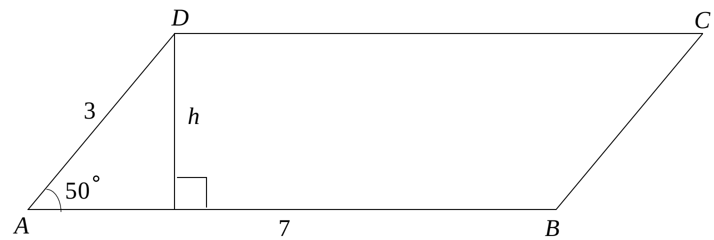
$$\begin{aligned} A' &= \frac{1}{2}\cdot br\cdot ar\cdot\sin\theta \\ &= r^2\left(\frac{1}{2}ba\cdot\sin\theta\right) \\ &= r^2 A, \end{aligned}$$

where A is the area of the original triangle.

Thus, scaling a triangle by a factor of r changes its area by a factor of r^2. This result agrees with the work you did in Investigation 4.17.

An Area Formula for Triangles *(continued)*

Problem 3 *(Student page 187)* Draw an altitude of length h from vertex D to side $\overline{AB}$:



You'll be finding the areas of more parallelograms in the next investigation.

Notice that

$$\sin 50^\circ = \frac{h}{3},$$

so

$$h = 3 \sin 50^\circ.$$

Since the area of a parallelogram is the product of its base times the height,

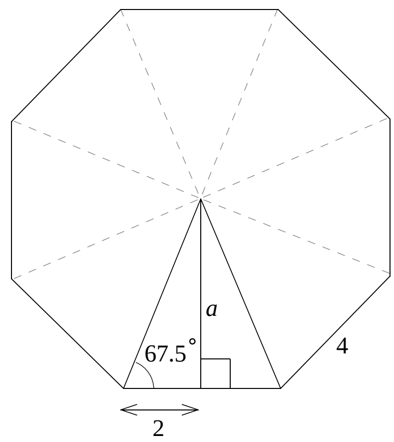
$$\text{Area } (ABCD) = 7 \cdot 3 \sin 50^\circ = 21 \sin 50^\circ \approx 16.09.$$

Problem 4 *(Student page 187)* Divide the octagon into eight congruent triangles by connecting the center to each vertex. Each triangle has a base of length 4 (the sidelength of the octagon) and a height of a, where a is the apothem of the octagon. First, let's solve for a.

Look at any of the eight triangles — it's isosceles, as each segment from the center of the octagon to a vertex has the same length. And its central angle is

$$\frac{360^\circ}{8} = 45^\circ.$$

This means that each base angle of the triangle measures 67.5°. The apothem of the triangle bisects the base (since the altitude from a vertex angle of an isosceles triangle is also a median), and divides the triangle into two right triangles. Focus on one of these right triangles, and use some trigonometry to calculate a.



You get

$$\tan 67.5^\circ = \frac{a}{2},$$

so

$$a = 2 \tan 67.5^\circ.$$

Thus, each of the eight triangles has an area of

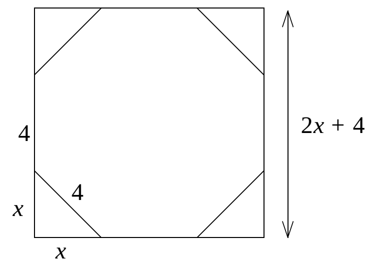
$$\frac{1}{2} \cdot 4 \cdot a = 4 \tan 67.5^\circ.$$

Since the octagon is composed of the eight triangles, its area is equal to

$$(8)(4 \tan 67.5^\circ) = 32 \tan 67.5^\circ \approx 77.25.$$

Here's another way to solve this problem: Draw the octagon with sidelength 4 inside a square, as shown in the following figure. This produces four congruent isosceles right triangles that are contained inside the square, but outside the octagon. If you knew the areas of the four triangles and the square, you could find the area of the octagon — just subtract the area of the triangles from the area of the square.

AN AREA FORMULA FOR TRIANGLES *(continued)*



Let x be the length of each leg of the four right triangles. Each hypotenuse is also a side of the octagon, and hence has length 4. By the Pythagorean Theorem,

$$\begin{aligned} x^2 + x^2 &= 4^2 \\ 2x^2 &= 16 \\ x^2 &= 8 \\ x &= 2\sqrt{2}. \end{aligned}$$

Since the area of each triangle is

$$\frac{1}{2}x^2,$$

each one has an area of 4, so the four triangles together have an area of 16.

Each side of the square has length $2x + 4$, which is equal to

$$4\sqrt{2} + 4.$$

Thus, the area of the square is

$$(4\sqrt{2} + 4)^2.$$

After a bit of algebra, you see that this is equal to

$$48 + 32\sqrt{2}.$$

Subtracting the area of the triangles from the area of the square gives the area of the octagon as

$$(48 + 32\sqrt{2}) - 16 = 32 + 32\sqrt{2},$$

which is, as before, approximately 77.25.

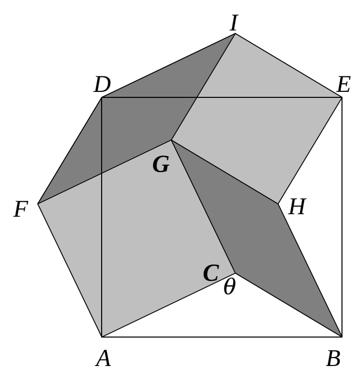
EXTENDING THE PYTHAGOREAN THEOREM

Problem 1 *(Student page 188)* In a triangle ABC, suppose that a, b, and c are the sidelengths opposite $\angle A$, $\angle B$, and $\angle C$, respectively. If you compare $a^2 + b^2$ with c^2, you will find that:

- If $m\angle C < 90^\circ$, then $a^2 + b^2$ is greater than c^2.
- If $m\angle C = 90^\circ$, then $a^2 + b^2$ is equal to c^2, as the Pythagorean Theorem states.
- If $m\angle C > 90^\circ$, then $a^2 + b^2$ is less than c^2.

Problem 2 *(Student page 189)* In the triangle on the left, angle C is obtuse, so you can predict that $a^2 + b^2$ will be less than c^2. In the triangle on the right, angle C is acute, so $a^2 + b^2$ should be greater than c^2.

Problem 3 *(Student page 190)* As you drag vertex C, notice that parallelograms $FDIG$ and $CGHB$ seem to remain congruent, as do the four triangles $\triangle ABC$, $\triangle DEI$, $\triangle ADF$, and $\triangle BEH$.



Let's show that the four triangles are indeed congruent.

Since $ADEB$ is a square, we have

$$\overline{AD} \cong \overline{BE} \cong \overline{DE} \cong \overline{AB}.$$

Notice that each of these segments is a side of one of the triangles. Since $AFGC$ is a square, its four sides are congruent. Furthermore, $\overline{FG} \cong \overline{DI}$ and $\overline{CG} \cong \overline{BH}$ since $FDIG$ and $CGHB$ are parallelograms. Combining this information, we have

$$\overline{AF} \cong \overline{BH} \cong \overline{DI} \cong \overline{AC}.$$

These segments form a second set of congruent sides of the four triangles. Applying similar reasoning to the square $GIEH$ tells us that we have a third set of congruent

sides of the four triangles:

$$\overline{FD} \cong \overline{HE} \cong \overline{IE} \cong \overline{CB}.$$

Thus, by the SSS test, all four triangles are congruent.

A note for purists: Actually, we've made several assumptions here that have not been proven. Why, for instance, when you construct parallelogram *FDIG* does point *D* land on a vertex of square *ADEB*? And why does point *E* on square *GIEH* also land on a vertex of *ADEB*?

Problem 4 *(Student page 190)* As point *C* moves, with θ always greater than 90°, the area of Region 1 remains equal to the area of Region 2.

Notice that if you place *C* on $\overline{AB}$, the four shaded quadrilaterals fit completely inside the large square.

Problem 5 *(Student page 191)* Cut out $\triangle ADF$ and place it on $\triangle BEH$. Also cut out $\triangle DEI$ and place it on $\triangle ABC$. Since these triangles are all congruent, everything will fit exactly. This rearranges Region 1 so that it fits completely inside Region 2.

Problem 6 *(Student page 191)* The visual equality says that the area of the large square is equal to the sum of the areas of the two small squares and the two parallelograms.

Problem 7 *(Student page 191)* Since $AB = c$, it follows that

$$\text{Area } (ADEB) = c^2.$$

Since $AC = b$, you know

$$\text{Area } (AFGC) = b^2.$$

Notice that $GH = CB = a$. Therefore,

$$\text{Area } (GIEH) = a^2.$$

Problem 8 *(Student page 191)* Parallelogram *CGHB* has two sides of length *a* and two sides of length *b*. Since $m\angle ACG = 90^\circ$, it follows that

$$m\angle GCB = 360^\circ - (\theta + 90^\circ) = 270^\circ - \theta.$$

Recall that opposite angles in a parallelogram are congruent, so $m\angle GHB$ is also equal to $270^\circ - \theta$. Since the sum of the angles in a quadrilateral is 360°, it follows that

$$2m\angle CBH = 360^\circ - [2(270^\circ - \theta)] = 2\theta - 180^\circ,$$

so

$$m\angle CBH = \theta - 90^\circ,$$

and this is the same measurement as $\angle CGH$.

Thus parallelogram *CGHB* has sides of length *a* and *b*, and angles of measure $270^\circ - \theta$ and $\theta - 90^\circ$. Since parallelogram *FDIG* is congruent to *CGHB*, its measurements are the same.

Problem 9 *(Student page 191)* The statement of the theorem simply equates the area of Region 1 with the area of Region 2.

Problem 10 *(Student page 192)* Applying the Pythagorean extension to $\triangle ABC$ gives:

$$(AB)^2 = 4^2 + 6^2 + \text{ twice the area of parallelogram } (CGHB).$$

So,

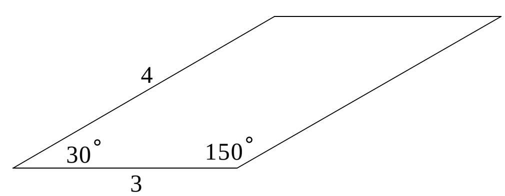
$$(AB)^2 = 4^2 + 6^2 + 2(12) = 76$$

and $AB = \sqrt{76}$.

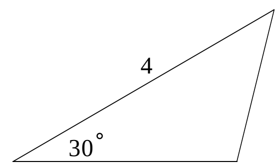
Problem 11 *(Student page 193)*

a. Applying the Pythagorean extension to $\triangle ABC$ gives

$$(AB)^2 = 4^2 + 3^2 + \text{ twice the area of the parallelogram below:}$$



The area of this parallelogram is twice the area of the triangle below:



From Investigation 4.23, you know that the area of this triangle is

$$\frac{1}{2} \cdot 3 \cdot 4 \cdot \sin 30^\circ = 3.$$

Thus, the area of the parallelogram is 6, and

$$(AB)^2 = 4^2 + 3^2 + 12$$

or

$$AB = \sqrt{37}.$$

b. Applying the method from part a gives

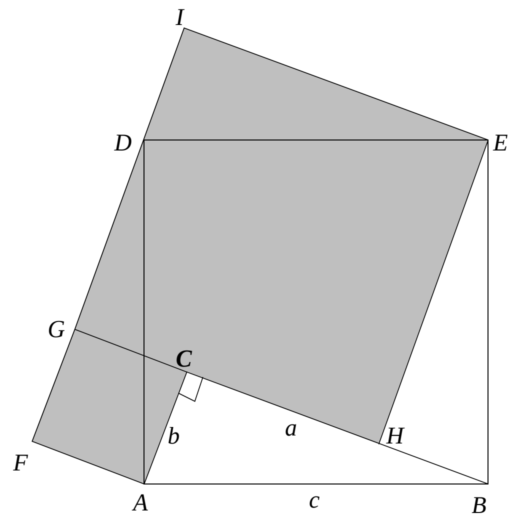
$$(AB)^2 = 41 + 40 \sin 10^\circ,$$

which means that AB is approximately 6.92.

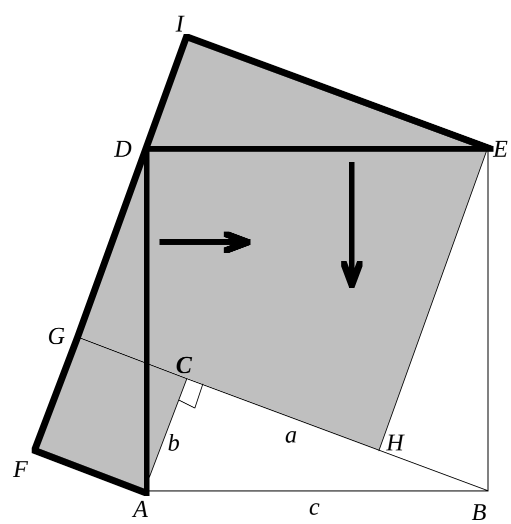
Problem 12 *(Student page 193)* The extension of the Pythagorean Theorem is more powerful because it applies to all triangles, not just right triangles.

EXTENDING THE PYTHAGOREAN THEOREM
(continued)

Problem 13 *(Student page 193)* When θ equals 90°, the two parallelograms disappear completely. The picture below shows what you get.



The shaded region consists of two squares: $AFGC$ with area b^2 and $GIEH$ with area a^2. This shaded region can fit exactly within $ADEB$. Simply place $\triangle DEI$ onto $\triangle ABC$ and $\triangle ADF$ onto $\triangle BEH$:

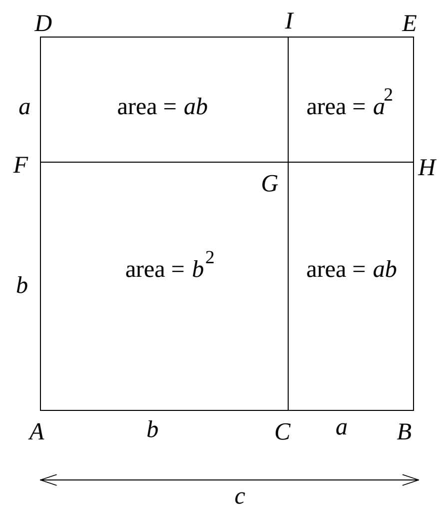


Thus the sum of the areas of the two smaller squares equals the area of the large square. In other words,

$$a^2 + b^2 = c^2.$$

This is the Pythagorean Theorem!

Problem 14 *(Student page 194)* When C lies along $\overline{AB}$, the two parallelograms become rectangles. You still have a square $AFGC$ with area b^2, and a square $GIEH$ with area a^2, but now you have two congruent rectangles, $CGHB$ and $FDIG$ with sidelengths a and b, and areas of ab. These four quadrilaterals fit completely inside the large square $ADEB$. (Notice that this large square has sidelength $c = a + b$.)

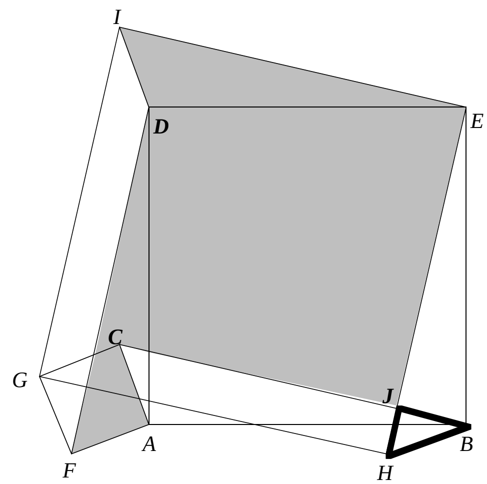


Since the large square has the same area as the sum of the four smaller quadrilaterals, you get the equation

$$(a + b)^2 = a^2 + b^2 + ab + ab = a^2 + 2ab + b^2.$$

This is a geometric proof of a well-known algebraic identity!

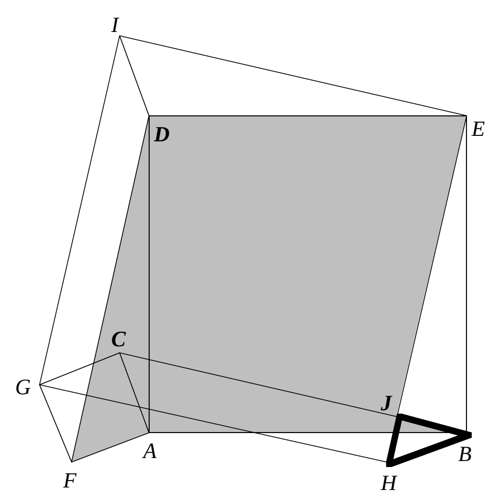
Problem 15 *(Student page 194)* After you've shaded in the areas of the squares and removed the areas of the parallelograms, you're left with:



The triangle *JHB* is highlighted to indicate that it should be subtracted from the shaded region.

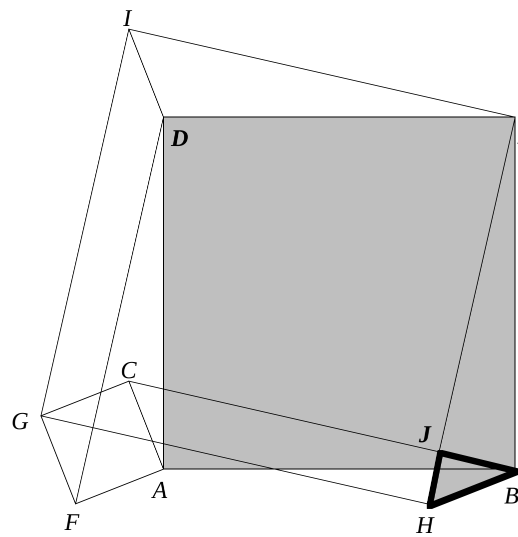
How can we now show that this all fits exactly into square *ADEB*?

First, move the shaded region of $\triangle DEI$ into $\triangle ABC$:



EXTENDING THE PYTHAGOREAN THEOREM
(continued)

Next, move the shaded region of $\triangle FDA$ into $\triangle HEB$:



In the picture above, we still have to remove the region enclosed by $\triangle JHB$. Be sure to notice that the part of $\triangle JHB$ that lies inside square *ADEB* has actually been counted twice (it was already shaded before you moved $\triangle FDA$). Once you've removed the region enclosed by $\triangle JHB$, you're left with *ADEB*. That's the end of the dissection argument and the end of the module!